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Daniel de Carvalho

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A Note on Weak Algebraic Theories

Daniel de Carvalho*

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Abstract: In our PhD thesis, we showed that for the study of denotational semantics of Linear Logic, it is crucial to generalize the standard notion of *monad* on a category. An earlier generalization was already given by Hoofman: *semi-monads*. But it was not suitable for our problem. This is why we introduced another generalization: *weak monads*. In this report, we present this new notion and give some examples.

Key-words: category theory, linear logic, denotational semantics.

* INRIA-Lorraine, Nancy – Daniel.decarvalho@loria.fr

Centre de recherche INRIA Nancy – Grand Est
LORIA, Technopôle de Nancy-Brabois, Campus scientifique,
615, rue du Jardin Botanique, BP 101, 54602 Villers-Lès-Nancy
Téléphone : +33 3 83 59 30 00 — Télécopie : +33 3 83 27 83 19

Une note sur les théories algébriques faibles

Résumé : Dans notre thèse de doctorat, nous avons montré que pour l'étude de la sémantique dénotationnelle de la logique linéaire, il est crucial de généraliser la notion standard de *monade* sur une catégorie. Auparavant, il existait déjà une généralisation de cette notion donnée par Hoffman : ce sont les *semi-monades*. Mais celle-ci ne permettait pas de résoudre notre problème. C'est pourquoi nous avons introduit une autre généralisation : les *monades faibles*. Dans ce rapport, nous présentons cette nouvelle notion et donnons quelques exemples.

Mots-clés : théorie des catégories, logique linéaire, sémantique dénotationnelle.

Notation

For any category $\mathbb{C}$, we denote by ob($\mathbb{C}$) the collection of objects of $\mathbb{C}$.

1 Introduction

The first works tackling the problem of giving a categorical interpretation of linear logic are those of Lafont ([10]) and Seely ([14]) - the reader can refer to [13] for a survey. The works of Benton, Bierman, Hyland and de Paiva ([1], [2] and [3]) led to the following axiomatics: a categorical model of the intuitionistic multiplicative exponential fragment of Linear Logic (IMELL) is a quadruple $(\mathcal{C}, \mathcal{L}, c, w)$ such that:

- $\mathcal{C} = (\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ is a closed symmetric monoidal category ;
- $\mathcal{L} = ((T, \mathsf{m}, \mathsf{n}), \delta, d)$ is a symmetric monoidal comonad on $\mathcal{C}$;
- c is a monoidal natural transformation from $(T, \mathsf{m}, \mathsf{n})$ to $\otimes \circ \Delta_{\mathcal{C}} \circ (T, \mathsf{m}, \mathsf{n})$ and w is a monoidal natural transformation from $(T, \mathsf{m}, \mathsf{n})$ to $*_{\mathcal{C}}$ such that
  - for any object A of the category $\mathbb{C}$, $((T(A), \delta_A), c_A, w_A)$ is a cocommutative comonoid in $(\mathbb{C}^{\mathbb{T}}, \otimes^{\mathbb{T}}, (I, \mathsf{n}), \alpha, \lambda, \rho)$
  - and for any $f \in \mathbb{C}^{\mathbb{T}}((TA, \delta_A), (TB, \delta_B))$, f is a comonoid morphism,

  where $\Delta_{\mathcal{C}}$ is the diagonal monoidal functor from $\mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$, $*_{\mathcal{C}}$ denotes the monoidal functor that sends any morphism to the identity on I, $\mathbb{T}$ is the comonad (T, δ, d) on $\mathbb{C}$ and $\mathbb{C}^{\mathbb{T}}$ is the category of $\mathbb{T}$-coalgebras.

In particular, (T, δ, d) is a comonad on the category $\mathbb{C}$. In such a model, if A is an atom, the axiom $!A \vdash !A$ and the η-expansion of the axiom are interpreted by the same morphism: the identity on $T(A)$, denoted by $id_{T(A)}$. But there are models of Linear Logic that don't satisfy this property. This is why we sought to weaken these axioms in order to capture these models. The starting point is the idea that we have to interpret the axiom by the identity on $T(A)$ and the η-expansion by $T(id_A)$ and that we don't ask to have $T(id_A) = id_{T(A)}$; in other words, T is not a functor anymore, but a *semi-functor*. Semi-functors have been introduced by Hayashi ([8]) in order to study the semantics of lambda calculus. In the same way, we introduced weakenings of the notions of comonads and monoidal comonads. In this paper, we present this weakening of the notion of comonad: *weak comonad*, i.e. *weak monad* in the opposite category.

We begin, in Section 2, to recall another generalization of the notion of monad given by Hoofman ([9]): the notion of *semi-monad*. The definition of weak monad is given in Section 3 and an equivalent definition in Section 4. We recall the definition of semi-adjunction in Section 4.1; it is worth noting that we have two dual statements (Proposition 6 and Proposition 7) that relate semi-monads and weak monads (see also the conclusion of section 3). In Section 5, we generalize the Eilenberg-Moore construction for weak monads and show that, in the same way as the standard construction gives rise to a resolution of a monad, our construction gives rise to a resolution of a weak-monad; this construction was used for giving our new axiomatics of models of Linear Logic presented in our PhD thesis. In Section 6, we give some examples of comonads on the category **Rel**. All these examples are provided by a general construction (Taylor's formula) that we present in Section 7. This construction comes from the study of differential nets introduced by Ehrhard and Régnier ([5]) and shows that differential nets can be seen as a decomposition of Linear Logic, in the same way as Linear Logic can be seen as a decomposition of Intuitionistic Logic.

2 Semi-monads

Definition 1 ([8]) *Let $\mathbb{C}$ and $\mathbb{D}$ be two categories. A* semi-functor $\mathbb{C} \to \mathbb{D}$ *is a pair (T_0, T_1) consisting of a function $T_0 : ob(\mathbb{C}) \to ob(\mathbb{D})$ and of a function T_1 that to each arrow $A \to B$ of the category $\mathbb{C}$ assigns an arrow $T_0(A) \to T_0(B)$ of the category $\mathbb{D}$ such that we have $T_1(g \circ f) = T_1(g) \circ T_1(f)$ whenever $g \circ f$ is defined.*

In other words, a semi-functor is a functor, except that it does not necessarily preserve identities.

Of course, we can still compose semi-functors as we compose functors and thus obtain a category whose objects are categories and whose arrows are semi-functors. We have also natural transformations between semi-functors and we can define vertical compositions $\bullet$ and horizontal compositions $\circ$ of natural transformations between semi-functors as we do it for natural transformations between functors.

- *If F, G and H are three semi-functors $\mathbb{C} \to \mathbb{D}$ and if σ is a natural transformation $F \Rightarrow G$ and τ is a natural transformation $G \Rightarrow H$, then $\tau \bullet \sigma$ is a natural transformation $F \Rightarrow H$ defined by $(\tau \bullet \sigma)_C = \tau_C \circ \sigma_C$.*
- *If F and G are two semi-functors $\mathbb{C} \to \mathbb{D}$ and F' and G' are two semi-functors $\mathbb{D} \to \mathbb{E}$ and if σ is a natural transformation $F \Rightarrow G$ and σ' is a natural transformation $F' \Rightarrow G'$, then $\sigma' \circ \sigma$ is a natural transformation $F' \circ F \Rightarrow G' \circ G$ defined by $(\sigma' \circ \sigma)_C = \sigma'_{G(C)} \circ F'(\sigma_C)$.*

Normal natural transformations *are what Hoofman, in [9], calls* semi-natural transformations*.*

Definition 2 *Let F and G be two semi-functors $\mathbb{C} \to \mathbb{D}$. A natural transformation $\alpha : F \Rightarrow G$ is said to be* normal *if, for any object C of the category $\mathbb{C}$, we have*

$$\alpha_C \circ F(id_C) = \alpha_C \quad .$$

Two remarks:

- *a natural transformation $F \Rightarrow G$ is normal if, and only if, for any object C of the category $\mathbb{C}$, we have*
$$G(id_C) \circ \alpha_C = \alpha_C \quad ;$$
- *if F or G is a functor, then α is a normal natural transformation from the functor F to the functor G.*

We already know that the category of categories and semi-functors does exist. Better, the following data define a 2-category $\mathbf{Cat}_s$:

- *the objects are categories ;*
- *for any objects $\mathbb{C}$ and $\mathbb{D}$, $\mathbf{Cat}_s(\mathbb{C}, \mathbb{D})$ is the category whose objects are semi-functors $\mathbb{C} \to \mathbb{D}$, whose arrows are normal natural transformations, whose composition is the vertical composition $\bullet$ and whose identity on a semi-functor F is $(F(id_C))_{C \in \mathbb{C}}$;*
- *composition is composition of the semi-functors and the horizontal composition $\circ$ of normal natural transformations ;*
- *the identity on an object $\mathbb{C}$ is the identity functor $id_{\mathbb{C}}$ on the category $\mathbb{C}$ and the identity natural transformation $id_{id_{\mathbb{C}}}$ from the natural transformation $id_{\mathbb{C}}$ to the natural transformation $id_{\mathbb{C}}$.*

Now, given a 2*-category* $\mathbf{C}$*, we can define what is a* monad in the 2-category $\mathbf{C}$ *as Street did in [15].*

Definition 3 ([15]) *Let* $\mathbf{C}$ *be a* 2*-category and let* $\mathbb{C}$ *be an object of the* 2*-category* $\mathbf{C}$*. A* monad in the 2-category $\mathbf{C}$ on $\mathbb{C}$ *is a triple* (T, μ, η) *consisting of*

- *an arrow* $T : \mathbb{C} \to \mathbb{C}$ *;*
- *a* 2*-arrow* $\eta : id_{\mathbb{C}} \Rightarrow T$ *;*
- *a* 2*-arrow* $\mu : T \circ T \Rightarrow T$

such that the diagrams

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "\mu \circ id_T"] \arrow[d, "id_T \circ \mu"'] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"'] & T \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} T \arrow[r, "\eta \circ id_T"] \arrow[dr, "id_T"'] & T \circ T \arrow[d, "\mu"] \\ & T \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} T \circ T \arrow[d, "\mu"'] & T \arrow[l, "T \circ \eta"'] \arrow[dl, "id_T"] \\ T & \end{tikzcd} \tag{3}$$

commute in the category $\mathbf{C}(\mathbb{C}, \mathbb{C})$*.*

We can now define the notion of semi-monad

Definition 4 *A* semi-monad on a category $\mathbb{C}$ *is a monad in the* 2*-category* $\mathbf{Cat}_s$ *on the category* $\mathbb{C}$*.*

The following proposition gives a characterization of the semi-monads on a category $\mathbb{C}$*. That is how [9] defines semi-(co)monads.*

Proposition 1 *The triple* (T, μ, η) *is a semi-monad on a category* $\mathbb{C}$ *if, and only if,*

- T *is a semi-functor* $\mathbb{C} \to \mathbb{C}$*,*
- η *is a natural transformation* $id_{\mathbb{C}} \Rightarrow T$*,*
- *and* μ *is a natural transformation* $T \circ T \Rightarrow T$

such that, for any object A of the category $\mathbb{C}$, the diagrams

$$\begin{tikzcd} T \circ T \circ T(A) \arrow[r, "\mu_{T(A)}"] \arrow[d, "T(\mu_A)"'] & T \circ T(A) \arrow[d, "\mu_A"] \\ T \circ T(A) \arrow[r, "\mu_A"'] & T(A) \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} T(A) \arrow[r, "\eta_{T(A)}"] \arrow[dr, "T(id_A)"'] & T \circ T(A) \arrow[d, "\mu_A"] \\ & T(A) \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} T \circ T(A) \arrow[d, "\mu_A"'] & T(A) \arrow[l, "T(\eta_A)"'] \arrow[dl, "T(id_A)"] \\ T(A) & \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} T \circ T(A) \arrow[r, "\mu_A"] \arrow[dr, "\mu_A"'] & T(A) \arrow[d, "T(id_A)"] \\ & T(A) \end{tikzcd} \tag{4}$$

commute in the category $\mathbb{C}$.

Proof

Diagrams (1), (2) and (3) correspond respectively to the diagrams (1), (2) and (3) of Definition 3. Diagram (4) means the normality of μ.

Diagram (1) means the associativity of the multiplication μ: we find exactly the same diagram in the definition of a monad. Diagrams (2) and (3) mean the neutrality of η for μ and correspond to the two other necessary usual diagrams in the definition of a monad ; but, now, we don't necessarily have $T(id_A) = id_{T(A)}$: here, we require that the composition of the two arrows be equal to $T(id_A)$. Lastly, we demand that Diagram (4) commutes: this one does obviously automatically hold in the case of a monad.

The notion of semi-monad is a good starting point in order to have a denotational semantics of the η-expanded proofs of Linear Logic. But how to interpret all *the proofs ? We need a comultiplication which* is not *normal. So, we want another weakening of the notion of monad with a non normal multiplication.*

But we have a problem to capture this notion: we fail to define a 2-category whose objects would be categories, whose arrows would be semi-functors and whose 2-arrows would be natural transformations. Indeed, now, the identity on a semi-functor $\mathbb{C} \to \mathbb{D}$ cannot be $(F(id_C))_{C \in \mathbb{C}}$ anymore, it has to be $(id_{F(C)})_{C \in \mathbb{C}}$. Now, the horizontal composition $\circ$ is not functorial anymore: if F is not a functor, then $id_F \circ id_{id_{\mathbb{C}}} \neq id_F$. Then ?

3 Weak monads

By considering Definition 3 again, we note that we could be able to define what is a monad in a 2-category in another way. Indeed, given a 2-category $\mathbf{C}$ and an object $\mathbb{C}$ of the 2-category $\mathbf{C}$, it is clear that the triple $(\mathbf{C}(\mathbb{C},\mathbb{C}),\circ,id_{\mathbb{C}})$ is a trict monoidal category. Then a monad in the 2-category $\mathbf{C}$ on the object $\mathbb{C}$ of the 2-category $\mathbf{C}$ is the same thing as a monoid in the strict monoidal category $(\mathbf{C}(\mathbb{C},\mathbb{C}),\circ,id_{\mathbb{C}})$.

Given a category $\mathbb{C}$, we denote by $\mathbf{1/2End}(\mathbb{C})$ the category whose objects are semi-functors $\mathbb{C}\to\mathbb{C}$, whose arrows are natural transformations, whose composition is the vertical composition $\bullet$ and whose identity on an object F is $(id_{F(C)})_{C\in\mathbb{C}}$. We already saw that $\circ$ is not a functor. Hence the triple $(\mathbf{1/2End}(\mathbb{C}),\circ,id_{\mathbb{C}})$ is not a strict monoidal category. But almost... It is an example of what we call a strict semi-monoidal category, *notion that generalizes the notion of strict monoidal category.*

Definition 5 *A* strict semi-monoidal category *is a triple $(\mathbb{C},\otimes,I)$ consisting of*

- *a category $\mathbb{C}$;*
- *a semi-functor $\otimes:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ such that*
  - *for any objects A, B and C of the category $\mathbb{C}$, we have $(A\otimes B)\otimes C = A\otimes(B\otimes C)$*
  - *and for any arrows f, g and h of the category $\mathbb{C}$, we have $(f\otimes g)\otimes h = f\otimes(g\otimes h)$;*
- *and an object I of the category $\mathbb{C}$ such that, for any object A of the category $\mathbb{C}$, we have $I\otimes A = A = A\otimes I$.*

In other words, a strict semi-monoidal category $(\mathbb{C},\otimes,I)$ is a strict monoidal category, except that we don't necessarily have $id_A\otimes id_B = id_{A\otimes B}$. There is nothing to change to the usual definition of strict monoidal functor.

Definition 6 *A* strict monoidal functor *$(F,\mathsf{m},\mathsf{n})$ from a strict semi-monoidal category $(\mathbb{C},\otimes,I)$ to a strict semi-monoidal category $(\mathbb{C}',\otimes',I')$ is a triple $(F,\mathsf{m},\mathsf{n})$ consisting of*

- *a functor $F:\mathbb{C}\to\mathbb{C}'$,*
- *a normal natural transformation $\mathsf{m}:\otimes'\circ(F\times F)\Rightarrow F\circ\otimes$,*
- *and an arrow $\mathsf{n}:I'\to F(I)$ of $\mathbb{C}'$*

such that, for any objects A, B and C of the category $\mathbb{C}$, the diagram

$$\begin{tikzcd} F(A)\otimes' F(B)\otimes' F(C) \arrow[r, "\mathsf{m}_{A,B}\otimes' id_{F(C)}"] \arrow[d, "id_{F(A)}\otimes'\mathsf{m}_{B,C}"'] & F(A\otimes B)\otimes' F(C) \arrow[d, "\mathsf{m}_{A\otimes B,C}"] \\ F(A)\otimes' F(B\otimes C) \arrow[r, "\mathsf{m}_{A,B\otimes C}"'] & F(A\otimes B\otimes C) \end{tikzcd}$$

commutes in the category $\mathbb{C}'$ and, for any object A of the category $\mathbb{C}$, the two diagrams

$$\begin{tikzcd} F(A) \arrow[r, "\mathsf{n} \otimes' id_{F(A)}"] \arrow[dr, "id_{I'} \otimes' id_{F(A)}"'] & F(I) \otimes' F(A) \arrow[d, "\mathsf{m}_{I,A}"] \\ & F(A) \end{tikzcd}$$

$$\begin{tikzcd} F(A) \otimes' F(I) \arrow[d, "\mathsf{m}_{A,I}"'] & F(A) \arrow[l, "id_{F(A)} \otimes' \mathsf{n}"'] \arrow[dl, "id_{F(A)} \otimes' id_{I'}"] \\ F(A) & \end{tikzcd}$$

commute in the category $\mathbb{C}'$.

Definition 7 *Let $(\mathbb{C}, \otimes, I)$, $(\mathbb{C}', \otimes', I')$ and $(\mathbb{C}'', \otimes'', I'')$ be three strict semi-monoidal categories. Let $(F, \mathsf{m}, \mathsf{n})$ be a strict monoidal functor from $(\mathbb{C}, \otimes, I)$ to $(\mathbb{C}', \otimes', I')$ and let $(F', \mathsf{m}', \mathsf{n}')$ be a strict monoidal functor from $(\mathbb{C}', \otimes', I')$ to $(\mathbb{C}'', \otimes'', I'')$. We set*

$$(F', \mathsf{m}', \mathsf{n}') \circ (F, \mathsf{m}, \mathsf{n}) = (F' \circ F, \mathsf{m}'', F'(\mathsf{n}) \circ \mathsf{n}') \ ,$$

where $\mathsf{m}'' = (\mathsf{m}''_{A,B})_{A,B \in ob(\mathbb{C})}$ with $\mathsf{m}''_{A,B} = F'(\mathsf{m}_{A,B}) \circ \mathsf{m}'_{F(A),F(B)}$.

It is clear that the following data define a category $\mathbf{1/2MonCat}$*:*

- *the objects are strict semi-monoidal categories ;*
- *an arrow $\mathcal{C} \to \mathcal{C}'$ is a strict monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$;*
- *the identity on the object $(\mathbb{C}, \otimes, I)$ is $(id_{\mathbb{C}}, (id_{A,B})_{A,B \in ob(\mathbb{C})}, id_I)$;*
- *composition is this defined in Definition 7.*

Note that the terminal object of the full subcategory **MonCat** *whose objects are strict monoidal categories is stil the terminal object of the category* $\mathbf{1/2MonCat}$*. The following definition generalizes the notion of* monoid *in a strict monoidal category.*

Definition 8 *A monoid in a strict semi-monoidal category $\mathcal{C}$ is a triple (A, μ, η) such that there exists a strict monoidal functor $(F, \mathsf{m}, \mathsf{n})$ from the terminal object $(\mathbf{1}, \otimes, *)$ of the category $\mathbf{1/2MonCat}$ to $\mathcal{C}$ such that $A = F(*)$, $\mu = \mathsf{m}_{*,*}$ and $\eta = \mathsf{n}$.*

The following proposition gives a characterization of the monoids in a strict semi-monoidal caregory.

Proposition 2 *The triple (A, μ, η) is a monoid in a strict semi-monoidal category $(\mathbb{C}, \otimes, I)$ if, and only if, A is an object of the category $\mathbb{C}$, μ is an arrow $A \otimes A \to A$ of the category $\mathbb{C}$ and η is an arrow $I \to A$ of the category $\mathbb{C}$ such that the diagrams*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes id_A"] \arrow[d, "id_A \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad (1)$$

$$\begin{tikzcd} A \arrow[r, "\eta \otimes id_A"] \arrow[dr, "id_I \oplus id_A"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} A \otimes A \arrow[d, "\mu"'] & A \arrow[l, "id_A \otimes \eta"'] \arrow[dl, "id_A \oplus id_I"] \\ A & \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} A \otimes A \arrow[r, "id_A \otimes id_A"] \arrow[dr, "\mu"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \tag{4}$$

commute in the category $\mathbb{C}$.

Proof

Simple check.

Diagrams (1), (2) and (3) hold for a monoid in a strict monoidal category. Diagram (4) automatically holds whenever the strict semi-monoidal category is indeed a strict monoidal category.

There is nothing to change to the usual definition of monoid morphism.

Definition 9 *Let* (A, μ, η) *and* (A', μ', η') *be two monoids in a strict semi-monoidal category* $(\mathbb{C}, \otimes, I)$*. A* monoid morphism *from* (A, μ, η) *to* (A', μ', η') *is an arrow* $\lambda : A \to A'$ *of the category* $\mathbb{C}$ *such that the two diagrams*

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "\lambda \otimes \lambda"'] & A \arrow[d, "\lambda"] \\ A' \otimes A' \arrow[r, "\mu'"'] & A' \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & A \arrow[d, "\lambda"] \\ & A' \end{tikzcd}$$

commute in the category $\mathbb{C}$.

It is clear that, for any semi-monoidal category $\mathcal{C} = (\mathbb{C}, \otimes, I)$, the following data define a category $\mathbf{Mon}(\mathcal{C})$*:*

- *the objects are monoids ;*
- *the arrows are monoid morphisms ;*
- *the identity on (A, μ, η) is the identity on A ;*
- *composition is the same as composition in the category $\mathbb{C}$.*

Definition 10 *Let $\mathbb{C}$ be a category. Set $\mathbf{WMon}(\mathbb{C}) = \mathbf{Mon}(\mathbf{1/2End}(\mathbb{C}), \circ, id_{\mathbb{C}})$. The objects of the category $\mathbf{WMon}(\mathbb{C})$ are called* weak monads on the category $\mathbb{C}$ *and the arrows of the category $\mathbf{WMon}(\mathbb{C})$ are called* monad morphisms.

The following proposition gives a characterization of the weak monads on a given category.

Proposition 3 *Let $\mathbb{C}$ be a category. A weak monad on a category $\mathbb{C}$ is a triple (T, μ, η) consisting of*

- *a semi-functor $T : \mathbb{C} \to \mathbb{C}$,*
- *a natural transformation $\eta : id_{\mathbb{C}} \Rightarrow T$,*
- *and a natural transformation $\mu : T \circ T \Rightarrow T$*

such that, for any object A of the category $\mathbb{C}$, the diagrams

$$\begin{tikzcd} T \circ T \circ T(A) \arrow[r, "\mu_{T(A)}"] \arrow[d, "T(\mu_A)"'] & T \circ T(A) \arrow[d, "\mu_A"] \\ T \circ T(A) \arrow[r, "\mu_A"'] & T(A) \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} T(A) \arrow[r, "\eta_{T(A)}"] \arrow[dr, "id_{T(A)}"'] & T \circ T(A) \arrow[d, "\mu_A"] \\ & T(A) \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} T \circ T(A) \arrow[d, "\mu_A"'] & T(A) \arrow[l, "T(\eta_A)"'] \arrow[dl, "T(id_A)"] \\ T(A) & \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} T \circ T(A) \arrow[r, "T(id_{T(A)})"] \arrow[dr, "\mu_A"'] & T \circ T(A) \arrow[d, "\mu_A"] \\ & T(A) \end{tikzcd} \tag{4}$$

commute in the category $\mathbb{C}$.

Proof

Simple check.

Diagram (1) means the associativity of μ: we find it in the definition of monad and of semi-monad. Diagrams (2) and (3) mean the neutrality of η for μ: we find Diagram (3) in the definition of semi-monad, but not Diagram (2). Diagram (4) automatically holds in a monad: be carefull, it is not the diagram that would mean the normality of μ.

Proposition 4 *Let (T, μ, η) be a weak monad on a category $\mathbb{C}$. Then the following assertions are equivalent:*

- *the triple (T, μ, η) is a semi-monad on the category $\mathbb{C}$;*
- *the natural transformation $\mu : T \circ T \Rightarrow T$ is normal ;*
- *the semi-functor $T : \mathbb{C} \to \mathbb{C}$ is a functor ;*
- *the weak monad (T, μ, η) on the category $\mathbb{C}$ is a monad.*

Proof

Simple check.

Every weak monad induces a semi-monad.

Definition 11 *Let $\mathbb{C}$ be a category. We denote by S_0 the function that to each object (T, μ, η) of the category $\mathbf{WMon}(\mathbb{C})$ assigns the triple $(T, (id_T \circ id_{\mathbb{C}}) \bullet \mu, \eta)$.*

Lemma 1 *Let $\mathbb{C}$ be a category. For any object $\mathbb{T}$ of the category $\mathbf{WMon}(\mathbb{C})$, $S_0(\mathbb{T})$ is an object of the category $\mathbf{1/2Mon}(\mathbb{C})$.*

Proof

Simple check.

Definition 12 *Let $\mathbb{C}$ be a category. We denote by S_1 the function that to each $\lambda \in \mathbf{WMon}(\mathbb{C})((T, \mu, \eta), (T', \mu', \eta'))$ assigns $(id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda$.*

Lemma 2 *Let $\mathbb{C}$ be a category. For any $\lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}, \mathbb{T}')$, we have $S_1(\lambda) \in \mathbf{1/2Mon}(\mathbb{C})(S_0(\mathbb{T}), S_0(\mathbb{T}'))$.*

Proof

Set $\mathbb{T} = (T, \mu, \eta)$ and $\mathbb{T}' = (T', \mu', \eta')$. We have

$$\begin{array}{rcl} S_1(\lambda) \bullet \eta & = & ((id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda) \bullet \eta \\ & & \quad \textit{(by definition of } S_1\textit{)} \\ & = & (id_{T'} \circ id_{\mathbb{C}}) \bullet (\lambda \bullet \eta) \\ & = & (id_{T'} \circ id_{\mathbb{C}}) \bullet \eta' \\ & & \quad \textit{(because } \lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}, \mathbb{T}')\textit{)} \\ & = & \eta' \ . \end{array}$$

We have

$$\begin{aligned}
S_1(\lambda) \bullet \mu &= ((id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda) \bullet \mu \\
&= (id_{T'} \circ id_{\mathbb{C}}) \bullet (\lambda \bullet \mu) \\
&= (id_{T'} \circ id_{\mathbb{C}}) \bullet (\mu' \bullet (\lambda \circ \lambda)) \\
&\quad \text{(because } \lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}, \mathbb{T}')\text{)} \\
&= ((id_{T'} \circ id_{\mathbb{C}}) \bullet \mu') \bullet (\lambda \circ \lambda) \\
&= (\mu' \bullet (id_{T' \circ T'} \circ id_{\mathbb{C}})) \bullet (\lambda \circ \lambda) \\
&= (\mu' \bullet (id_{T'} \circ id_{\mathbb{C}} \circ id_{\mathbb{T}'} \circ id_{\mathbb{C}})) \bullet (\lambda \circ \lambda) \\
&= \mu' \bullet ((id_{T'} \circ id_{\mathbb{C}} \circ id_{\mathbb{T}'} \circ id_{\mathbb{C}}) \bullet (\lambda \circ \lambda)) \\
&= \mu' \bullet (((id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda) \circ ((id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda)) \\
&= \mu' \bullet (S_1(\lambda) \circ S_1(\lambda)) \ .
\end{aligned}$$

Lemma 3 *Let $\mathbb{C}$ be a category. For any $\lambda \in \mathbf{WMon}(\mathbb{C})((T, \mu, \eta), (T', \mu', \eta'))$ and any $\lambda' \in \mathbf{WMon}(\mathbb{C})((T', \mu', \eta'), (T'', \mu'', \eta''))$, we have $S_1(\lambda' \bullet \lambda) = S_1(\lambda') \bullet S_1(\lambda)$.*

Proof

$$\begin{aligned}
S_1(\lambda') \bullet S_1(\lambda) &= ((id_{T''} \circ id_{\mathbb{C}}) \bullet \lambda') \bullet ((id_{T'} \circ id_{\mathbb{C}}) \bullet \lambda) \\
&\quad \text{(by definition of } S_1\text{)} \\
&= (id_{T''} \circ id_{\mathbb{C}}) \bullet (\lambda' \bullet (id_{T'} \circ id_{\mathbb{C}})) \bullet \lambda \\
&= (id_{T''} \circ id_{\mathbb{C}}) \bullet ((id_{T''} \circ id_{\mathbb{C}}) \bullet \lambda') \bullet \lambda \\
&= ((id_{T''} \circ id_{\mathbb{C}}) \bullet (id_{T''} \circ id_{\mathbb{C}})) \bullet (\lambda' \bullet \lambda) \\
&= ((id_{T''} \bullet id_{T''}) \circ (id_{\mathbb{C}} \circ id_{\mathbb{C}})) \bullet (\lambda' \bullet \lambda) \\
&= (id_{T''} \bullet id_{T''}) \bullet (\lambda' \bullet \lambda) \\
&= S_1(\lambda' \bullet \lambda) \ .
\end{aligned}$$

Lemmas 1, 2 and 3 show that, for any category $\mathbb{C}$, we can define a functor $S = (S_0, S_1) : \mathbf{WMon}(\mathbb{C}) \to \mathbf{1/2Mon}(\mathbb{C})$ with S_0 defined as in Definition 11 and with S_1 defined as in Definition 12.

4 Weak monads in extension form

Let $\mathbb{C}$ be a category.

It is well-known that the notion of monad *on the category $\mathbb{C}$ is equivalent to the following notion (see, for instance, [12] that calls it* algebraic theory in extension form*).*

Definition 13 *A* monad in extension form *on the category $\mathbb{C}$ is a triple $(T, \eta, -^*)$, where*

- *T is a function $ob(\mathbb{C}) \to ob(\mathbb{C})$;*
- *η is a function that to each object A of $\mathbb{C}$ assigns an arrow $\eta_A : A \to T(A)$ in $\mathbb{C}$;*
- *$-^*$ is a function that to each $f \in \mathbb{C}(A, T(B))$ assigns an element f^* of $\mathbb{C}(T(A), T(B))$*

such that

(i) for any $f \in \mathbb{C}(A, T(B))$*, we have* $f^* \circ \eta_A = f$*;*

(ii) for any $f \in \mathbb{C}(A, T(B))$ *and for any* $g \in \mathbb{C}(B, T(C))$*, we have* $g^* \circ f^* = (g^* \circ f)^*$*;*

(iii) for any object A *of the category* $\mathbb{C}$*,* ${\eta_A}^* = id_{T(A)}$*.*

The aim of this section is to show that the notion of weak monad *is equivalent to the following notion (see Proposition 5).*

Definition 14 *A* weak monad in extension form *on the category* $\mathbb{C}$ *is a triple* $(T, \eta, -^*)$*, where*

- T *is a function* $ob(\mathbb{C}) \to ob(\mathbb{C})$*;*
- η *is a function that to each object* A *of* $\mathbb{C}$ *assigns an arrow* $\eta_A : A \to T(A)$ *in* $\mathbb{C}$*;*
- $-^*$ *is a function that to each* $f \in \mathbb{C}(A, T(B))$ *assigns an element* f^* *of* $\mathbb{C}(T(A), T(B))$

such that

(i) for any $f \in \mathbb{C}(A, T(B))$*, we have* $f^* \circ \eta_A = f$*;*

(ii) for any $f \in \mathbb{C}(A, T(B))$ *and for any* $g \in \mathbb{C}(B, T(C))$*, we have* $g^* \circ f^* = (g^* \circ f)^*$*.*

Definition 15 *Let* $\mathbb{T} = (T, \eta, -^*)$ *and* $\mathbb{T}' = (T', \eta', -^{*'})$ *be two weak monads in extension form on the category* $\mathbb{C}$*. A* monad morphism *from* $\mathbb{T}$ *to* $\mathbb{T}'$ *is a function* λ *that to each object* A *of the category* $\mathbb{C}$ *assigns an element* λ_A *of* $\mathbb{C}(T(A), T'(A))$ *such that*

(i) for any object A *of the category* $\mathbb{C}$*, we have* $\lambda_A \circ \eta_A = \eta'_A$*;*

(ii) for any $f \in \mathbb{C}(A, T(B))$*, we have* $\lambda_B \circ f^* = (\lambda_B \circ f)^{*'} \circ \lambda_A$*.*

Definition 16 *Let* $\mathbb{T} = (T, \eta, -^*)$*,* $\mathbb{T}' = (T', \eta', -^{*'})$ *and* $\mathbb{T}'' = (T'', \eta'', -^{*''})$ *be three weak monads in extension form on the category* $\mathbb{C}$*. Let* λ *be a monad morphism from* $\mathbb{T}$ *to* $\mathbb{T}'$ *and let* λ' *be a monad morphism from* $\mathbb{T}'$ *to* $\mathbb{T}''$*. We denote by* $\lambda' \bullet \lambda$ *the function that to each object* A *of the category* $\mathbb{C}$ *assigns the element* $(\lambda' \bullet \lambda)_A = \lambda'_A \circ \lambda_A$ *of* $\mathbb{C}(T(A), T''(A))$*.*

Lemma 4 *Let* $\mathbb{T} = (T, \eta, -^*)$*,* $\mathbb{T}' = (T', \eta', -^{*'})$ *and* $\mathbb{T}'' = (T'', \eta'', -^{*''})$ *be three weak monads in extension form on the category* $\mathbb{C}$*. Let* λ *be a monad morphism from* $\mathbb{T}$ *to* $\mathbb{T}'$ *and let* λ' *be a monad morphism from* $\mathbb{T}'$ *to* $\mathbb{T}''$*. Then* $\lambda' \bullet \lambda$ *is a monad morphism from* $\mathbb{T}$ *to* $\mathbb{T}''$*.*

Proof

- *For any object* A *of the category* $\mathbb{C}$*, we have*

$$
\begin{aligned}
(\lambda' \bullet \lambda)_A \circ \eta_A &= (\lambda'_A \circ \lambda_A) \circ \eta_A \\
&= \lambda'_A \circ (\lambda_A \circ \eta_A) \\
&= \lambda'_A \circ {\eta'}_A \\
&\qquad \textit{(because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}'\textit{)} \\
&= {\eta''}_A \\
&\qquad \textit{(because } \lambda' \textit{ is a monad morphism from } \mathbb{T}' \textit{ to } \mathbb{T}''\textit{)}\ .
\end{aligned}
$$

- *For any* $f \in \mathbb{C}(A, T(B))$, *we have*

$$\begin{array}{rcl} (\lambda' \bullet \lambda)_B \circ f^* & = & (\lambda'_B \circ \lambda_B) \circ f^* \\ & = & \lambda'_B \circ (\lambda_B \circ f^*) \\ & = & \lambda'_B \circ ((\lambda_B \circ f)^{*'} \circ \lambda_A) \\ & & \text{(because } \lambda \text{ is a monad morphism from } \mathbb{T} \text{ to } \mathbb{T}'\text{)} \\ & = & (\lambda'_B \circ (\lambda_B \circ f)^{*'}) \circ \lambda_A \\ & = & ((\lambda'_B \circ (\lambda_B \circ f))^{*''} \circ \lambda'_A) \circ \lambda_A \\ & = & ((\lambda'_B \circ \lambda_B) \circ f)^{*''} \circ (\lambda'_A \circ \lambda_A) \\ & = & ((\lambda' \bullet \lambda)_B \circ f)^{*''} \circ (\lambda' \bullet \lambda)_A \ . \end{array}$$

Definition 17 *Let* $\mathbb{T} = (T, \eta, -^*)$ *be a weak monad in extension form on the category* $\mathbb{C}$. *We denote by* $id_{\mathbb{T}}$ *the function that to each object* A *of the category* $\mathbb{C}$ *assigns* $id_{T(A)} \in \mathbb{C}(T(A), T(A))$.

Lemma 5 *Let* $\mathbb{T} = (T, \eta, -^*)$ *be a weak monad in extension form on the category* $\mathbb{C}$. *Then* $id_{\mathbb{T}}$ *is a monad morphism from* $\mathbb{T}$ *to* $\mathbb{T}$.

Proof

Trivial.

Lemma 6 *Let* $\mathbb{T} = (T, \eta, -^*)$ *and* $\mathbb{T}' = (T', \eta', -^{*'})$ *be two weak monads in extension form on the category* $\mathbb{C}$ *and let* λ *be a monad morphism from* $\mathbb{T}$ *to* $\mathbb{T}'$. *Then we have*

(i) $id_{\mathbb{T}'} \bullet \lambda = \lambda$;

(ii) $\lambda \bullet id_{\mathbb{T}} = \lambda$.

Proof

Trivial.

Lemmas 4, 5 and 6 show that we can define the category $\mathbf{eWMon}(\mathbb{C})$ *whose objects are weak monads in extension form, whose arrows* $\mathbb{T} \to \mathbb{T}'$ *are monad morphisms from* $\mathbb{T}$ *to* $\mathbb{T}'$, *whose composition is* $\bullet$ *and whose identity on* $\mathbb{T}$ *is* $id_{\mathbb{T}}$.

Definition 18 *We denote by* M_0 *the function that to each object* $\mathbb{T} = (T_0, \eta, -^*)$ *of the category* $\mathbf{eWMon}(\mathbb{C})$ *assigns the triple* (T, μ, η), *where*

- $T = (T_0, T_1)$, *where* T_1 *is a function that to each* $f \in \mathbb{C}(A, B)$ *assigns* $(\eta_B \circ f)^*$.
- μ *is a function that to each object* A *of the category* $\mathbf{WMon}(\mathbb{C})$ *assigns* $\mu_A = (id_{T(A)})^*$.

Lemma 7 *For any object* $\mathbb{T}$ *of the category* $\mathbf{eWMon}(\mathbb{C})$, $M_0(\mathbb{T})$ *is an object of the category* $\mathbf{WMon}(\mathbb{C})$.

Proof

Let $\mathbb{T} = (T_0, \eta, -^*)$ *be an object of the category* $\mathbf{eWMon}(\mathbb{C})$*. Set* $M_0(\mathbb{T}) = (T, \mu, \eta)$ *with* $T = (T_0, T_1)$*. Let* $f \in \mathbb{C}(A, B)$ *and* $g \in \mathbb{C}(B, C)$*. We have*

$$\begin{array}{rcl} T_1(g \circ f) & = & (\eta_C \circ (g \circ f))^* \\ & & \quad \textit{(by definition of } T_1\textit{)} \\ & = & ((\eta_C \circ g) \circ f)^* \\ & = & (((\eta_C \circ g)^* \circ \eta_B) \circ f)^* \\ & = & ((\eta_C \circ g)^* \circ (\eta_B \circ f))^* \\ & = & (\eta_C \circ g)^* \circ (\eta_B \circ f)^* \\ & = & T_1(g) \circ T_1(f) \ . \end{array}$$

Hence T *is a semi-functor* $\mathbb{C} \to \mathbb{C}$.

For any $f \in \mathbb{C}(A, B)$*, we have*

$$\begin{array}{rcl} \mu_B \circ (T \circ T)(f) & = & id^*_{T(B)} \circ (\eta_{T(B)} \circ (\eta_B \circ f)^*)^* \\ & = & (id^*_{T(B)} \circ (\eta_{T(B)} \circ (\eta_B \circ f)^*))^* \\ & = & ((id^*_{T(B)} \circ \eta_{T(B)}) \circ (\eta_B \circ f)^*)^* \\ & = & (id_{T(B)} \circ (\eta_B \circ f)^*)^* \\ & = & (\eta_B \circ f)^{**} \\ & = & ((\eta_B \circ f)^* \circ id_{T(A)})^* \\ & = & (\eta_B \circ f)^* \circ id^*_{T(A)} \\ & = & T(f) \circ \mu_A \\ & & \quad \textit{(by definition of } T \textit{ and } \mu\textit{)} \ . \end{array}$$

Hence μ *is a natural transformation* $T \circ T \Rightarrow T$.

For any $f \in \mathbb{C}(A, B)$*, we have*

$$\begin{array}{rcl} T(f) \circ \eta_A & = & (\eta_B \circ f)^* \circ \eta_A \\ & & \quad \textit{(by definition of } T\textit{)} \\ & = & \eta_B \circ f \ . \end{array}$$

Hence η *is a natural transformation* $id_{\mathbb{C}} \Rightarrow T$.

For any object A *of the category* $\mathbb{C}$*, we have*

$$\begin{array}{rcl} \mu_A \circ T(\mu_A) & = & id^*_{T(A)} \circ (\eta_{T(A)} \circ id^*_{T(A)})^* \\ & & \quad \textit{(by definition of } T \textit{ and } \mu\textit{)} \\ & = & (id^*_{T(A)} \circ (\eta_{T(A)} \circ id^*_{T(A)}))^* \\ & = & ((id^*_{T(A)} \circ \eta_{T(A)}) \circ id^*_{T(A)})^* \\ & = & (id_{T(A)} \circ id^*_{T(A)})^* \\ & = & {id^*_{T(A)}}^* \\ & = & (id^*_{T(A)} \circ id_{T(T(A))})^* \\ & = & id^*_{T(A)} \circ id^*_{T(T(A))} \\ & = & \mu_A \circ \mu_{T(A)} \ . \end{array}$$

For any object A *of the category* $\mathbb{C}$*, we have*

$$\begin{array}{rcl} \mu_A \circ \eta_{T(A)} & = & id^*_{T(A)} \circ \eta_{T(A)} \\ & & \quad \textit{(by definition of } \mu_A\textit{)} \end{array}$$

$$= \quad id_{T(A)} \ .$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{aligned}
\mu_A \circ T(\eta_A) &= id^*_{T(A)} \circ (\eta_{T(A)} \circ \eta_A)^* \\
&\quad \textit{(by definition of } \mu \textit{ and } T\textit{)} \\
&= (id^*_{T(A)} \circ (\eta_{T(A)} \circ \eta_A))^* \\
&= ((id^*_{T(A)} \circ \eta_{T(A)}) \circ \eta_A)^* \\
&= (id_{T(A)} \circ \eta_A)^* \\
&= (\eta_A \circ id_A)^* \\
&= T(id_A) \ .
\end{aligned}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{aligned}
\mu_A \circ T(id_{T(A)}) &= id^*_{T(A)} \circ (\eta_{T(A)} \circ id_{T(A)})^* \\
&\quad \textit{(by definition of } T \textit{ and } \mu\textit{)} \\
&= id^*_{T(A)} \circ \eta^*_{T(A)} \\
&= (id^*_{T(A)} \circ \eta_{T(A)})^* \\
&= id^*_{T(A)} \\
&= \mu_A \ .
\end{aligned}$$

Lemma 8 *Let $\mathbb{T}$ and $\mathbb{T}'$ be two objects of the category $\mathbf{eWMon}(\mathbb{C})$. Let λ be a monad morphism from $\mathbb{T}$ to $\mathbb{T}'$. Then we have $\lambda \in \mathbf{WMon}(\mathbb{C})(M_0(\mathbb{T}), M_0(\mathbb{T}'))$.*

Proof

Set $\mathbb{T} = (T_0, \eta, -^*)$, $\mathbb{T}' = (T'_0, \eta', -^{*'})$, $M_0(\mathbb{T}) = (T, \mu, \eta)$ and $M_0(\mathbb{T}') = (T', \mu', \eta')$.

- *For any $f \in \mathbb{C}(A, B)$, we have*

$$\begin{aligned}
T'(f) \circ \lambda_A &= (\eta'_B \circ f)^{*'} \circ \lambda_A \\
&= (\lambda_B \circ \eta_B \circ f)^{*'} \circ \lambda_A \\
&\quad \textit{(because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}'\textit{)} \\
&= \Lambda_B \circ (\eta_B \circ f)^* \\
&\quad \textit{(again because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}'\textit{)} \\
&= \lambda_B \circ T(f) \ .
\end{aligned}$$

- *For any object A of the category $\mathbb{C}$, we have*

$$\begin{aligned}
(\lambda \bullet \mu)_A &= \lambda_A \circ \mu_A \\
&= \lambda_A \circ {id_{T(A)}}^* \\
&= (\lambda_A \circ id_{T(A)})^{*'} \circ \lambda_{T(A)} \\
&\quad \textit{(because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}'\textit{)} \\
&= ({id_{T'(A)}}^{*'} \circ (\eta'_{T'(A)} \circ \lambda_A)^{*'}) \circ \lambda_{T(A)} \\
&= ({id_{T'(A)}}^{*'} \circ T'(\lambda_A)) \circ \lambda_{T(A)} \\
&= {id_{T'(A)}}^{*'} \circ (T'(\lambda_A) \circ \lambda_{T(A)}) \\
&= \mu'_A \circ (\lambda \circ \lambda)_A \\
&= (\mu' \bullet (\lambda \circ \lambda))_A \ .
\end{aligned}$$

- *For any object A of the category $\mathbb{C}$, we have*

$$\begin{array}{rcl} (\lambda \bullet \eta)_A & = & \lambda_A \circ \eta_A \\ & = & \eta'_A \\ & & \text{(because } \lambda \text{ is a monad morphism from } \mathbb{T} \text{ to } \mathbb{T}') \ . \end{array}$$

Lemmas 7 and 8 show that we can define a functor $M : \mathbf{eWMon}(\mathbb{C}) \to \mathbf{WMon}(\mathbb{C})$ with $M = (M_0, M_1)$, where M_0 is defined as in Definition 18 and M_1 is defined as follows: if $\lambda \in \mathbf{eWMon}(\mathbb{T}, \mathbb{T}')$, then $M_1(\lambda) = \lambda \in \mathbf{WMon}(M_0(\mathbb{T}), M_0(\mathbb{T}'))$.

Definition 19 *We denote by e_0 the function that to each object $\mathbb{T} = ((T_0, T_1), \mu, \eta)$ of the category $\mathbf{WMon}(\mathbb{C})$ assigns the triple $(T_0, \eta, -^*)$, where*

- *$-^*$ is a function that to each $f \in \mathbb{C}(A, T(B))$ assigns $\mu_B \circ T_1(f)$.*

Lemma 9 *For any object $\mathbb{T}$ of the category $\mathbf{WMon}(\mathbb{C})$, $e_0(\mathbb{T})$ is an object of the category $\mathbf{eWMon}(\mathbb{C})$.*

Proof

Let $\mathbb{T} = (T, \mu, \eta)$ be an object of the category $\mathbf{WMon}(\mathbb{C})$. We set $e_0(\mathbb{T}) = (T_0, \eta, -^*)$.
For any $f \in \mathbb{C}(A, T(B))$, we have

$$\begin{array}{rcl} f^* \circ \eta_A & = & (\mu_B \circ T(f)) \circ \eta_A \\ & & \quad \text{(by definition of } -^*) \\ & = & \mu_B \circ (T(f) \circ \eta_A) \\ & = & \mu_B \circ (\eta_B \circ f) \\ & = & (\mu_B \circ \eta_B) \circ f \\ & = & id_{T(B)} \circ f \\ & = & f \ . \end{array}$$

For any $f \in \mathbb{C}(A, T(B))$ and for any $g \in \mathbb{C}(B, T(C))$, we have

$$\begin{array}{rcl} g^* \circ f^* & = & (\mu_C \circ T(g)) \circ (\mu_B \circ T(f)) \\ & & \quad \text{(by definition of } -^*) \\ & = & \mu_C \circ (T(g) \circ \mu_B) \circ T(f) \\ & = & \mu_C \circ (\mu_{T(C)} \circ (T \circ T)(g)) \circ T(f) \\ & = & ((\mu_C \circ \mu_{T(C)}) \circ (T \circ T)(g)) \circ T(f) \\ & = & ((\mu_C \circ T(\mu_C)) \circ T(T(g))) \circ T(f) \\ & = & \mu_C \circ ((T(\mu_C) \circ T(T(g))) \circ T(f)) \\ & = & \mu_C \circ T(\mu_C \circ T(g)) \circ T(f) \\ & = & \mu_C \circ (T(g^*) \circ T(f)) \\ & = & \mu_C \circ T(g^* \circ f) \\ & = & (g^* \circ f)^* \ . \end{array}$$

Lemma 10 *Let $\mathbb{T}$ and $\mathbb{T}'$ be two objects of the category $\mathbf{WMon}(\mathbb{C})$. Let λ be a monad morphism from $\mathbb{T}$ to $\mathbb{T}'$. Then we have $\lambda \in \mathbf{eWMon}(\mathbb{C})(e_0(\mathbb{T}), e_0(\mathbb{T}'))$.*

Proof

Set $\mathbb{T} = (T, \mu, \eta)$, $\mathbb{T}' = (T', \mu', \eta')$, $e_0(\mathbb{T}) = (T_0, \eta, -^*)$ *and* $e_0(\mathbb{T}') = (T'_0, \eta', -^{*'})$.

For any object A *of the category* $\mathbb{C}$, *we have*

$$\begin{array}{rcl} \lambda_A \circ \eta_A & = & (\lambda \bullet \eta)_A \\ & = & \lambda'_A \\ & & \textit{(because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}') \ . \end{array}$$

For any $f \in \mathbb{C}(A, T(B))$, *we have*

$$\begin{array}{rcl} \lambda_B \circ f^* & = & \lambda_B \circ (\mu_B \circ T(f)) \\ & = & (\lambda_B \circ \mu_B) \circ T(f) \\ & = & (\lambda \bullet \mu)_B \circ T(f) \\ & = & (\mu' \bullet (\lambda \circ \lambda))_B \circ T(f) \\ & & \textit{(because } \lambda \textit{ is a monad morphism from } \mathbb{T} \textit{ to } \mathbb{T}') \\ & = & (\mu'_B \circ (\lambda \circ \lambda)_B) \circ T(f) \\ & = & \mu'_B \circ ((\lambda \circ \lambda)_B \circ T(f)) \\ & = & \mu'_B \circ ((T'(\lambda_B) \circ \lambda_{T(B)}) \circ T(f)) \\ & = & \mu'_B \circ (T'(\lambda_B) \circ (\lambda_{T(B)} \circ T(f))) \\ & = & \mu'_B \circ (T'(\lambda_B) \circ (T'(f) \circ \lambda_A)) \\ & = & \mu'_B \circ ((T'(\lambda_B) \circ T'(f)) \circ \lambda_A) \\ & = & \mu'_B \circ (T'(\lambda_B \circ f) \circ \lambda_A) \\ & = & (\mu'_B \circ T'(\lambda_B \circ f)) \circ \lambda_A \\ & = & (\lambda_B \circ f)^{*'} \circ \lambda_A \ . \end{array}$$

Lemmas 9 and 10 show that we can define a functor $e = (e_0, e_1) : \mathbf{WMon}(\mathbb{C}) \to \mathbf{eWMon}(\mathbb{C})$ *with* e_0 *defined as in Definition 19 and with* e_1 *defined as follows: if* $\lambda \in \mathbf{WMon}(\mathbb{T}, \mathbb{T}')$, *then* $e_1(\lambda) = \lambda \in \mathbf{eWMon}(e_0(\mathbb{T}), e_0(\mathbb{T}'))$.

Proposition 5 *The functors* M *and* e *are two isomorphisms of categories.*

Proof

Let $\mathbb{T} = ((T_0, T_1), \mu, \eta)$ *be an object of* $\mathbf{WMon}(\mathbb{C})$. *Set* $e(\mathbb{T}) = (T_0, \eta, -^*)$ *and* $M_0(e_0(\mathbb{T})) = ((T'_0, T'_1), \mu', \eta)$.

For any $f \in \mathbb{C}(A, B)$, *we have*

$$\begin{array}{rcl} T'_1(f) & = & (\eta_B \circ f)^* \\ & = & \mu_B \circ T_1(\eta_B \circ f) \\ & = & \mu_B \circ (T_1(\eta_B) \circ T_1(f)) \\ & = & (\mu_B \circ T_1(\eta_B)) \circ T_1(f) \\ & = & T_1(id_B) \circ T_1(f) \\ & & \textit{(because } ((T_0, T_1), \mu, \eta) \textit{ is an object of } \mathbf{WMon}(\mathbb{C})) \\ & = & T_1(id_B \circ f) \\ & = & T_1(f) \ . \end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rcl} \mu'_A & = & id^*_{T(A)} \\ & = & \mu_A \circ T_1(id_{T(A)}) \\ & = & \mu_A \\ & & \quad \text{(because } ((T_0,T_1),\mu,\eta) \text{ is an object of } \mathbf{WMon}(\mathbb{C})\text{)} \ . \end{array}$$

Hence $M \circ e = id_{\mathbf{WMon}}$.

Let $\mathbb{T} = (T_0, \eta, -^*)$ *be an object of* $\mathbf{eWMon}(\mathbb{C})$. *Set* $M_0(\mathbb{T}) = ((T_0, T_1), \mu, \eta)$ *and* $e_0(M_0(\mathbb{T})) = (T_0, \eta, -^{*'})$.

For any $f \in \mathbb{C}(A, T(B))$, *we have*

$$\begin{array}{rcl} f^{*'} & = & \mu_B \circ T_1(f) \\ & = & {id_{T(B)}}^* \circ (\eta_B \circ f)^* \\ & = & ({id_{T(B)}}^* \circ (\eta_B \circ f))^* \\ & = & (({id_{T(B)}}^* \circ \eta_B) \circ f)^* \\ & = & (id_{T(B)} \circ f)^* \\ & & \quad \text{(because } (T_0,\eta,-^*) \text{ is an object of } \mathbf{eWMon}(\mathbb{C})\text{)} \\ & = & f^* \ . \end{array}$$

Hence $e \circ M = id_{\mathbf{eWMon}}$.

Hayashi introduced in [8] the notion of semi-adjunction. *What is the relation between the semi-monads and the weak monads on the one hand and the semi-adjunctions on the other hand ?*

4.1 Semi-adjunctions

Definition 20 *[8] Let $\mathbb{C}$ and $\mathbb{D}$ be two categories. A semi-adjunction $\mathbb{C} \to \mathbb{D}$ is a quadruple (F, G, α, β), where F is a semi-functor $\mathbb{C} \to \mathbb{D}$, G is a semi-functor $\mathbb{D} \to \mathbb{C}$, α and β are two functions that to each $(A, B) \in ob(\mathbb{C}) \times ob(\mathbb{D})$ assign respectively a function $\alpha_{A,B} : \mathbb{D}(F(A), B) \to \mathbb{C}(A, G(B))$ and a function $\beta_{A,B} : \mathbb{C}(A, G(B)) \to \mathbb{D}(F(A), B)$ such that, for any $f \in \mathbb{D}(B, B')$ and for any $g \in \mathbb{C}(A', A)$, the diagrams*

$$\begin{tikzcd} \mathbb{D}(F(A),B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "f \circ - \circ F(g)"'] & \mathbb{C}(A,G(B)) \arrow[d, "G(f) \circ - \circ g"] \\ \mathbb{D}(F(A'),B') \arrow[r, "\alpha_{A',B'}"'] & \mathbb{C}(A',G(B')) \end{tikzcd} \quad (1)$$

$$\begin{tikzcd} \mathbb{D}(F(A),B) \arrow[d, "f \circ - \circ F(g)"'] & \mathbb{C}(A,G(B)) \arrow[l, "\beta_{A,B}"'] \arrow[d, "G(f) \circ - \circ g"] \\ \mathbb{D}(F(A'),B') \arrow[r, "\alpha_{A',B'}"'] & \mathbb{C}(A',G(B')) \end{tikzcd} \quad (2)$$

$$
\begin{tikzcd}
\mathbb{D}(F(A), B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "f \circ - \circ F(g)"'] & \mathbb{C}(A, G(B)) \arrow[d, "G(f) \circ - \circ g"] \\
\mathbb{D}(F(A'), B') & \mathbb{C}(A', G(B')) \arrow[l, "\beta_{A',B'}"]
\end{tikzcd}
\tag{3}
$$

$$
\begin{tikzcd}
\mathbb{D}(F(A), B) \arrow[d, "f \circ - \circ F(g)"'] & \mathbb{C}(A, G(B)) \arrow[l, "\beta_{A,B}"'] \arrow[d, "G(f) \circ - \circ g"] \\
\mathbb{D}(F(A'), B') & \mathbb{C}(A', G(B')) \arrow[l, "\beta_{A',B'}"]
\end{tikzcd}
\tag{4}
$$

commute.

Definition 21 *A semi-adjunction $(F, G, \alpha, \beta) : \mathbb{C} \to \mathbb{D}$ is said to be* normal *if, and only if,*

(i) for any $f \in \mathbb{D}(F(A), B)$, we have $\alpha_{A,B}(f \circ F(id_A)) = \alpha_{A,B}(f)$;

(ii) and for any $g \in \mathbb{C}(A, G(B))$, we have $\beta_{A,B}(G(id_B) \circ g) = \beta_{A,B}(g)$.

Remark 1 *Let (F, G, α, β) be a semi-adjunction $\mathbb{C} \to \mathbb{D}$. For any $f \in \mathbb{D}(F(A), B)$, we have $\alpha_{A,B}(f \circ F(id_A)) = G(id_B) \circ \alpha_{A,B}(f)$ and for any $g \in \mathbb{C}(A, G(B))$, we have $\beta_{A,B}(G(id_B) \circ g) = \beta_{A,B}(g) \circ F(id_A)$. Hence if F or G is a a functor, then the semi-adjunction $(F, G, \alpha, \beta) : \mathbb{C} \to \mathbb{D}$ is normal.*

Lemma 11 *Let (F, G, α, β) be a normal semi-adjunction $\mathbb{C} \to \mathbb{D}$.*

(i) For any $f \in \mathbb{D}(F(A), B)$, we have $\beta_{A,B} \circ \alpha_{A,B}(f) = f \circ F(id_A)$.

(ii) For any $g \in \mathbb{C}(A, G(B))$, we have $\alpha_{A,B} \circ \beta_{A,B}(g) = G(id_B) \circ g$.

Proof

(i) Let $f \in \mathbb{D}(F(A), B)$. We have

$$
\begin{array}{rcl}
\beta_{A,B} \circ \alpha_{A,B}(f) & = & \beta_{A,B}(\alpha_{A,B}(f \circ F(id_A))) \\
& & \quad \textit{(because the semi-adjunction is normal)} \\
& = & \beta_{A,B}(G(id_B) \circ \alpha_{A,B}(f) \circ id_A) \\
& & \quad \textit{(by Diagram (1) of Definition 20)} \\
& = & id_B \circ f \circ F(id_A) \\
& & \quad \textit{(by Diagram (3) of Definition 20)} \\
& = & f \circ F(id_A) \ .
\end{array}
$$

(ii) Let $g \in \mathbb{C}(A, G(B))$. We have

$$
\begin{array}{rcl}
\alpha_{A,B} \circ \beta_{A,B}(g) & = & \alpha_{A,B}(\beta_{A,B}(G(id_B) \circ g)) \\
& & \quad \textit{(because the semi-adjunction is normal)}
\end{array}
$$

$$\begin{aligned}
&= \alpha_{A,B}(id_B \circ \beta_{A,B}(g) \circ F(id_A)) \\
&\qquad \textit{(by Diagram (4) of Definition 20)} \\
&= G(id_B) \circ g \circ id_A \\
&\qquad \textit{(by Diagram (2) of Definition 20)} \\
&= G(id_B) \circ g \ .
\end{aligned}$$

Lemma 12 *Let (F, G, α, β) be a normal semi-adjunction $\mathbb{C} \to \mathbb{D}$.*

(i) For any $f \in \mathbb{D}(F(A), B)$ and for any $h \in \mathbb{C}(A', A)$, we have $\alpha_{A',B}(f \circ F(h)) = \alpha_{A,B}(f) \circ h$.

(ii) For any $f \in \mathbb{D}(F(A), B)$ and for any $k \in \mathbb{D}(B, B')$, we have $\alpha_{A,B'}(k \circ f) = G(k) \circ \alpha_{A,B}(f)$.

(iii) For any $g \in \mathbb{C}(A, G(B))$ and for any $h \in \mathbb{C}(A', A)$, we have $\beta_{A',B}(g \circ h) = \beta_{A,B}(g) \circ F(h)$.

(iv) For any $g \in \mathbb{C}(A, G(B))$ and for any $k \in \mathbb{D}(B, B')$, we have $\beta_{A,B'}(G(k) \circ g) = k \circ \beta_{A,B}(g)$.

Proof

(i) Let $f \in \mathbb{D}(F(A), B)$ and $h \in \mathbb{C}(A', A)$. We have

$$\begin{aligned}
\alpha_{A',B}(f \circ F(h)) &= \alpha_{A',B}(id_B \circ f \circ F(h)) \\
&= G(id_B) \circ \alpha_{A,B}(f) \circ h \\
&\qquad \textit{(by Diagram (1) of Definition 20)} \\
&= G(id_B) \circ \alpha_{A,B}(f) \circ id_A \circ h \\
&= \alpha_{A,B}(id_B \circ f \circ F(id_A)) \circ h \\
&\qquad \textit{(by Diagram (1) of Definition 20)} \\
&= \alpha_{A,B}(f) \circ h \ .
\end{aligned}$$

(ii) Let $f \in \mathbb{D}(F(A), B)$ and $k \in \mathbb{D}(B, B')$. We have

$$\begin{aligned}
\alpha_{A,B'}(k \circ f) &= \alpha_{A,B'}(k \circ f \circ F(id_A)) \\
&\qquad \textit{(because the semi-adjunction is normal)} \\
&= G(k) \circ \alpha_{A,B}(f) \circ id_A \\
&\qquad \textit{(by Diagram (1) of Definition 20)} \\
&= G(k) \circ \alpha_{A,B}(f) \ .
\end{aligned}$$

(iii) Let $g \in \mathbb{C}(A, G(B))$ and $h \in \mathbb{C}(A', A)$. We have

$$\begin{aligned}
\beta_{A',B}(g \circ h) &= \beta_{A',B}(G(id_B) \circ g \circ h) \\
&\qquad \textit{(because the semi-adjunction is normal)} \\
&= id_B \circ \beta_{A,B}(g) \circ F(h) \\
&\qquad \textit{(by Diagram (4) of Definition 20)} \\
&= \beta_{A,B}(g) \circ F(h) \ .
\end{aligned}$$

(iv) Let $g \in \mathbb{C}(A, G(B))$ and $k \in \mathbb{D}(B, B')$. We have

$$\begin{array}{rcl}\beta_{A,B'}(G(k) \circ g) & = & \beta_{A,B'}(G(k) \circ g \circ id_A) \\ & = & k \circ \beta_{A,B}(g) \circ F(id_A) \\ & & \quad \textit{(by Diagram (4) of Definition 20)} \\ & = & k \circ id_B \circ \beta_{A,B}(g) \circ F(id_A) \\ & = & k \circ \beta_{A,B}(G(id_B) \circ g \circ id_A) \\ & & \quad \textit{(by Diagram (4) of Definition 20)} \\ & = & k \circ \beta_{A,B}(g) \ .\end{array}$$

Proposition 6 *Let (F, G, α, β) be a semi-adjunction $\mathbb{C} \to \mathbb{D}$ such that F is a functor $\mathbb{C} \to \mathbb{D}$. We set $T = G \circ F$. For any object A of the category $\mathbb{C}$, we set $\eta_A = \alpha_{A,F(A)}(id_{F(A)})$ and $\mu_A = G(\beta_{T(A),F(A)}(id_{T(A)}))$. Then the triple (T, μ, η) is a semi-monad on the category $\mathbb{C}$.*

Proof

For any $f \in \mathbb{C}(A, A')$, we have

$$\begin{array}{rcl}T(f) \circ \eta_A & = & G(F(f)) \circ \alpha_{A,F(A)}(id_{F(A)}) \\ & & \quad \textit{(by Lemma 12 (ii))} \\ & = & \alpha_{A,F(A')}(F(f) \circ id_{F(A)}) \\ & = & \alpha_{A,F(A')}(id_{F(A')} \circ F(f)) \\ & & \quad \textit{(by Lemma 12 (i))} \\ & = & \alpha_{A',F(A')}(id_{F(A')}) \circ f \\ & = & \eta_{A'} \circ f \ .\end{array}$$

For any $f \in \mathbb{C}(A, A')$, we have

$$\begin{array}{rcl}T(f) \circ \mu_A & = & G(F(f)) \circ G(\beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(F(f) \circ \beta_{T(A),F(A)}(id_{T(A)})) \\ & & \quad \textit{(by Lemma 12 (iv))} \\ & = & G(\beta_{T(A),F(A')}(G(F(f)) \circ id_{T(A)})) \\ & = & G(\beta_{T(A),F(A')}(id_{T(A')} \circ T(f))) \\ & & \quad \textit{(by Lemma 12 (iii))} \\ & = & G(\beta_{T(A'),F(A')}(id_{T(A')}) \circ F(T(f))) \\ & = & G(\beta_{T(A'),F(A')}(id_{T(A')})) \circ G(F(T(f))) \\ & = & \mu_{A'} \circ (T \circ T(f)) \ .\end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rcl}T(id_A) \circ \mu_A & = & G \circ F(id_A) \circ G(\beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(F(id_A) \circ \beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(id_{F(A)} \circ \beta_{T(A),F(A)}(id_{T(A)})) \\ & & \quad \textit{(because } F \textit{ is a functor)} \\ & = & G(\beta_{T(A),F(A)}(id_{T(A)})) \\ & = & \mu_A \ .\end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rl} & \beta_{T(A),F(A)}(id_{T(A)}) \circ (F \circ G)(\beta_{T(A),F(A)}(id_{T(A)})) \\ = & \beta_{T(T(A)),F(A)}(id_{T(A)} \circ G(\beta_{T(A),F(A)}(id_{T(A)}))) \\ & \textit{(by Lemma 12 (iii))} \\ = & \beta_{T(T(A)),F(A)}(G(\beta_{T(A),F(A)}(id_{T(A)})) \circ id_{T(T(A))}) \\ = & \beta_{T(A),F(A)}(id_{T(A)}) \circ \beta_{T(T(A)),F(T(A))}(id_{T(T(A))}) \\ & \textit{(by Lemma 12 (iv))} \end{array}$$

hence

$$\mu_A \circ T(\mu_A) = \mu_A \circ \mu_{T(A)} \ .$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rcl} \mu_A \circ T(\eta_A) & = & G(\beta_{T(A),F(A)}(id_{T(A)})) \circ (G \circ F)(\alpha_{A,F(A)}(id_{F(A)})) \\ & = & G(\beta_{T(A),F(A)}(id_{T(A)}) \circ F(\alpha_{A,F(A)}(id_{F(A)}))) \\ & & \textit{(by Lemma 12 (iii))} \\ & = & G(\beta_{A,F(A)}(id_{T(A)} \circ \alpha_{A,F(A)}(id_{F(A)}))) \\ & = & G(\beta_{A,F(A)} \circ \alpha_{A,F(A)}(id_{F(A)})) \\ & & \textit{(by Lemma 11 (i))} \\ & = & G(id_{F(A)} \circ F(id_A)) \\ & = & T(id_A) \ . \end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rcl} \mu_A \circ \eta_{T(A)} & = & G(\beta_{T(A),F(A)}(id_{T(A)})) \circ \alpha_{T(A),F(T(A))}(id_{F(T(A))}) \\ & & \textit{(by Lemma 12 (ii))} \\ & = & \alpha_{T(A),F(A)}(\beta_{T(A),F(A)}(id_{T(A)}) \circ id_{F(T(A))}) \\ & = & \alpha_{T(A),F(A)}(\beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(id_{F(A)}) \circ id_{T(A)} \\ & & \textit{(by Lemma 11 (ii))} \\ & = & G(id_{F(A)}) \\ & = & G \circ F(id_A) \\ & & \textit{(because } F \textit{ is a functor)} \\ & = & T(id_A) \ . \end{array}$$

Hence, when we have a semi-adjunction between two semi-functors and the left semi-adjoint is a functor, then we have an induced semi-monad by the semi-adjunction.

Proposition 7 *Let (F, G, α, β) be a semi-adjunction $\mathbb{C} \to \mathbb{D}$ such that G is a functor $\mathbb{D} \to \mathbb{C}$. We set $T = G \circ F$. For any object A of the category $\mathbb{C}$, we set $\eta_A = \alpha_{A,F(A)}(id_{F(A)})$ and $\mu_A = G(\beta_{T(A),F(A)}(id_{T(A)}))$. Then the triple (T, μ, η) is a weak monad on the category $\mathbb{C}$.*

Proof

For any $f \in \mathbb{C}(A, A')$, we have

$$\begin{array}{rcl} T(f) \circ \eta_A & = & G(F(f)) \circ \alpha_{A,F(A)}(id_{F(A)}) \\ & = & \alpha_{A,F(A')}(F(f) \circ id_{F(A)}) \\ & & \textit{(by Lemma 12 (ii))} \\ & = & \alpha_{A,F(A')}(id_{F(A')} \circ F(f)) \\ & = & \alpha_{A',F(A')}(id_{F(A')}) \circ f \\ & & \textit{(by Lemma 12 (i))} \\ & = & \eta_{A'} \circ f \ . \end{array}$$

For any $f \in \mathbb{C}(A, A')$, we have

$$\begin{array}{rcl} T(f) \circ \mu_A & = & G(F(f)) \circ G(\beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(F(f) \circ \beta_{T(A),F(A)}(id_{T(A)})) \\ & = & G(\beta_{T(A),F(A')}(G(F(f)) \circ id_{T(A)})) \\ & & \textit{(by Lemma 12 (iv))} \\ & = & G(\beta_{T(A),F(A')}(id_{T(A')} \circ T(f))) \\ & = & G(\beta_{T(A'),F(A')}(id_{T(A')}) \circ F(T(f))) \\ & & \textit{(by Lemma 12 (iii))} \\ & = & G(\beta_{T(A'),F(A')}(id_{T(A')})) \circ G(F(T(f))) \\ & = & \mu_{A'} \circ (T \circ T(f)) \ . \end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rl} & \beta_{T(A),F(A)}(id_{T(A)}) \circ (F \circ G)(\beta_{T(A),F(A)}(id_{T(A)})) \\ = & \beta_{T(T(A)),F(A)}(id_{T(A)} \circ G(\beta_{T(A),F(A)}(id_{T(A)}))) \\ & \textit{(by Lemma 12 (iii))} \\ = & \beta_{T(T(A)),F(A)}(G(\beta_{T(A),F(A)}(id_{T(A)})) \circ id_{T(T(A))}) \\ = & \beta_{T(A),F(A)}(id_{T(A)}) \circ \beta_{T(T(A)),F(T(A))}(id_{T(T(A))}) \\ & \textit{(by Lemma 12 (iv))} \end{array}$$

hence

$$\mu_A \circ T(\mu_A) = \mu_A \circ \mu_{T(A)} \ .$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{array}{rcl} \mu_A \circ T(\eta_A) & = & G(\beta_{T(A),F(A)}(id_{T(A)})) \circ (G \circ F)(\alpha_{A,F(A)}(id_{F(A)})) \\ & = & G(\beta_{T(A),F(A)}(id_{T(A)}) \circ F(\alpha_{A,F(A)}(id_{F(A)}))) \\ & = & G(\beta_{A,F(A)}(id_{T(A)} \circ \alpha_{A,F(A)}(id_{F(A)}))) \\ & & \textit{(by Lemma 12 (iii))} \\ & = & G(\beta_{A,F(A)} \circ \alpha_{A,F(A)}(id_{F(A)})) \\ & & \textit{(by Lemma 11 (i))} \\ & = & G(id_{F(A)} \circ F(id_A)) \\ & = & T(id_A) \ . \end{array}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{aligned}
\mu_A \circ \eta_{T(A)} &= G(\beta_{T(A),F(A)}(id_{T(A)})) \circ \alpha_{T(A),F(T(A))}(id_{F(T(A))}) \\
&= \alpha_{T(A),F(A)}(\beta_{T(A),F(A)}(id_{T(A)}) \circ id_{F(T(A))}) \\
&\quad \text{(by Lemma 12 (ii))} \\
&= \alpha_{T(A),F(A)}(\beta_{T(A),F(A)}(id_{T(A)})) \\
&\quad \text{(by Lemma 11 (ii))} \\
&= G(id_{F(A)}) \circ id_{T(A)} \\
&= G(id_{F(A)}) \\
&\quad \text{(because } G \text{ is a functor)} \\
&= id_{T(A)} \ .
\end{aligned}$$

For any object A of the category $\mathbb{C}$, we have

$$\begin{aligned}
\mu_A \circ T(id_{T(A)}) &= G(\beta_{T(A),F(A)}(id_{T(A)})) \circ (G \circ F)(id_{T(A)}) \\
&= G(\beta_{T(A),F(A)}(id_{T(A)}) \circ F(id_{T(A)})) \\
&\quad \text{(by Lemma 12 (iii))} \\
&= G(\beta_{T(A),F(A)}(id_{T(A)})) \\
&= \mu_A \ .
\end{aligned}$$

Hence when have a semi-adjunction between two semi-functors and the right semi-adjoint is a functor, we have an induced weak monad by the semi-adjunction.

Conversely, we may ask whether each weak monad originates in this way from a semi-adjunction in which the right semi-functor is a functor. More formally, we ask if each weak monad has a resolution in the following sense.

Definition 22 *Given a weak monad $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbb{C}$, a* resolution of the weak monad $\mathbb{T}$ *is a semi-adjunction $(F, G, \alpha, \beta) : \mathbb{C} \to \mathbb{D}$ such that*

- *G is a functor $\mathbb{D} \to \mathbb{C}$;*
- *$T = G \circ F$;*
- *for any $A \in ob(\mathbb{C})$, we have $\eta_A = \alpha_{A,F(A)}(id_{F(A)})$;*
- *for any $A \in ob(\mathbb{C})$, we have $\mu_A = G(\beta_{T(A),F(A)}(id_{T(A)})$.*

This definition generalizes the definition of resolution of a monad *that can be found in [11]:*

Given a monad $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbb{C}$, a resolution *of the monad $\mathbb{T}$ is a quadruple $(\mathbb{D}, G, F, \epsilon)$ such that*

- *$\mathbb{D}$ is a category;*
- *$T = G \circ F$;*
- *the quadruple (F, G, η, ϵ) is an adjunction $\mathbb{C} \to \mathbb{D}$;*
- *and $\mu = id_U \circ \epsilon \circ id_F$.*

Indeed, if we have a resolution $(F, G, \alpha, \beta) : \mathbb{C} \to \mathbb{D}$ of a monad $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbb{C}$ in the sense of Definition 22, then $(\mathbb{D}, G, F, \epsilon)$ with $\epsilon = (\epsilon_B)_{B \in ob(\mathbb{D})}$ and $\epsilon_B = \beta_{G(B),B}(id_{G(B)})$ is a resolution of the monad $\mathbb{T}$ in the sense of [11].

The next section shows that a generalization of the Eilenberg-Moore construction gives rise to resolutions of weak monads.

5 Algebras for a weak monad

We generalize the Eilenberg-Moore construction. Let $\mathbb{C}$ be a category.

Definition 23 *Let $\mathbb{T} = (T, \mu, \eta)$ be an object of the category* **WMon**. *The following data define a category $\mathbb{C}^{\mathbb{T}}$:*

- *the objects are of the shape (A, h), where A is an object of the category $\mathbb{C}$ and $h \in \mathbb{C}(T(A), A)$ such that the diagrams*

$$\begin{tikzcd} T \circ T(A) \arrow[r, "T(h)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "h"] \\ T(A) \arrow[r, "h"'] & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & T(A) \arrow[d, "h"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} T(A) \arrow[r, "T(id_A)"] \arrow[dr, "h"'] & T(A) \arrow[d, "h"] \\ & A \end{tikzcd}$$

  commute in the category $\mathbb{C}$;

- *an arrow $(A, h) \rightarrow (A', h')$ is an arrow $f : A \rightarrow A'$ in the category $\mathbb{C}$ such that the diagram*

$$\begin{tikzcd} A \arrow[d, "f"'] & T(A) \arrow[l, "h"'] \arrow[d, "T(f)"] \\ A' & T(A') \arrow[l, "h'"] \end{tikzcd}$$

  commute in the category $\mathbb{C}$;

- *for any object (A, h), the identity on (A, h) is the identity on A;*

- *composition is the same as composition in the category $\mathbb{C}$.*

Definition 24 *For any object $\mathbb{T}$ of the category* **WMon**$(\mathbb{C})$, *we denote by $G^{\mathbb{T}}$ the functor $\mathbb{C}^{\mathbb{T}} \rightarrow \mathbb{C}$ that to each object (A, h) of the category $\mathbb{C}^{\mathbb{T}}$ assigns the object A of the category $\mathbb{C}$ and that to each arrow f of the category $\mathbb{C}^{\mathbb{T}}$ assigns the arrow f of the category $\mathbb{C}$.*

Definition 25 *We denote by* $\mathbf{Con}(\mathbb{C})$ *the full subcategory of the slice category* $\mathbf{Cat}/\mathbb{C}$ *whose objects are faithful functors* $\mathbb{D} \to \mathbb{C}$ *for any object* $\mathbb{D}$ *of the category* $\mathbf{Cat}$*, where* $\mathbf{Cat}$ *denotes the category of categories.*

Definition 26 *We denote by* EM_0 *the function that to each object* $\mathbb{T}$ *of the category* $\mathbf{WMon}(\mathbb{C})$ *assigns the functor* $G^{\mathbb{T}} : \mathbb{C}^{\mathbb{T}} \to \mathbb{C}$.

Lemma 13 *For any object* $\mathbb{T}$ *of the category* $\mathbf{WMon}(\mathbb{C})$*,* $EM_0(\mathbb{T})$ *is an object of the category* $\mathbf{Con}(\mathbb{C})$.

PROOF

Trivial.

Lemma 14 *For any* $\lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})$*, if* (A, h) *is an object of the category* $\mathbb{C}^{\mathbb{T}}$*, then* $(A, h \circ \lambda_A)$ *is an object of the category* $\mathbb{C}^{\mathbb{T}'}$.

PROOF

Set $\mathbb{T} = (T, \mu, \eta)$ *and* $\mathbb{T}' = (T', \mu', \eta')$*. We have*

$$\begin{array}{rcl}
(h \circ \lambda_A) \circ \mu'_A & = & h \circ (\lambda_A \circ \mu'_A) \\
& = & h \circ (\lambda \bullet \mu')_A \\
& = & h \circ (\mu \bullet (\lambda \circ \lambda))_A \\
& & \quad \textit{(because } \lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})\textit{)} \\
& = & h \circ (\mu_A \circ (\lambda \circ \lambda)_A) \\
& = & (h \circ T(h)) \circ (\lambda_{T(A)} \circ T'(\lambda_A)) \\
& = & h \circ (T(h) \circ \lambda_{T(A)}) \circ T'(\lambda_A) \\
& = & h \circ (\lambda_A \circ T'(h)) \circ T'(\lambda_A) \\
& = & (h \circ \lambda_A) \circ (T'(h) \circ T'(\lambda_A)) \\
& = & (h \circ \lambda_A) \circ T'(h \circ \lambda_A) \quad .
\end{array}$$

We have

$$\begin{array}{rcl}
(h \circ \lambda_A) \circ \eta'_A & = & h \circ (\lambda_A \circ \eta'_A) \\
& = & h \circ (\lambda \bullet \eta')_A \\
& = & h \circ \eta_A \\
& & \quad \textit{(because } \lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})\textit{)} \\
& = & id_A \quad .
\end{array}$$

We have

$$\begin{array}{rcl}
(h \circ \lambda_A) \circ T'(id_A) & = & h \circ (\lambda_A \circ T'(id_A)) \\
& = & h \circ (T(id_A) \circ \lambda_A) \\
& = & (h \circ T(id_A)) \circ \lambda_A \\
& = & h \circ \lambda_A \quad .
\end{array}$$

Lemma 15 *For any* $\lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})$*, if* $f \in \mathbb{C}^{\mathbb{T}}((A, h), (A', h'))$*, then* $f \in \mathbb{C}^{\mathbb{T}'}((A, h \circ \lambda_A), (A', h' \circ \lambda_{A'}))$.

Proof

We have

$$\begin{array}{rcl} f \circ (h \circ \lambda_A) & = & (f \circ h) \circ \lambda_A \\ & = & (h' \circ T(f)) \circ \lambda_A \\ & = & h' \circ (T(f) \circ \lambda_A) \\ & = & h' \circ (\lambda_{A'} \circ T'(f)) \\ & & \text{(because } \lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})\text{)} \\ & = & (h' \circ \lambda_{A'}) \circ T'(f) \ . \end{array}$$

By Lemmas 14 and 15, to each $\lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})$*, we can define a functor* $EM_1(\lambda) = (Q_0, Q_1) : \mathbb{C}^{\mathbb{T}} \to \mathbb{C}^{\mathbb{T}'}$ *as follows:*

- Q_0 *is the function that to each object* (A, h) *of the category* $\mathbb{C}^{\mathbb{T}}$ *assigns the pair* $(A, h \circ \lambda_A)$*;*
- Q_1 *is the function that to each* $f \in \mathbb{C}^{\mathbb{T}}((A, h), (A', h'))$ *assigns* $f \in \mathbb{C}^{\mathbb{T}'}((A, h \circ \lambda_A), (A', h' \circ \lambda_{A'}))$.

And it is obvious that we have $G^{\mathbb{T}'} \circ EM_1(\lambda) = G^{\mathbb{T}}$*. Lastly, if* $\lambda \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}', \mathbb{T})$ *and* $\lambda' \in \mathbf{WMon}(\mathbb{C})(\mathbb{T}'', \mathbb{T}')$*, then for any object* (A, h) *of the category* $\mathbb{C}^{\mathbb{T}}$*, we have*

$$\begin{array}{rcl} EM_1(\lambda')(EM_1(\lambda)(A, h)) & = & EM_1(\lambda')(A, h \circ \lambda_A) \\ & = & (A, h \circ \lambda_A \circ \lambda'_A) \\ & = & (A, h \circ (\lambda \bullet \lambda')_A) \\ & & \text{(by definition of } \bullet\text{)} \\ & = & EM_1(\lambda \bullet \lambda')(A, h) \ . \end{array}$$

Hence $EM = (EM_0, EM_1)$ *is a functor* $\mathbf{WMon}(\mathbb{C})^{op} \to \mathbf{Con}(\mathbb{C})$.

Definition 27 *For any object* $\mathbb{T}$ *of the category* $\mathbf{WMon}(\mathbb{C})$*, we denote by* $F^{\mathbb{T}}$ *the functor* $\mathbb{C} \to \mathbb{C}^{\mathbb{T}}$ *that to each object* A *of the category* $\mathbb{C}$ *assigns the object* $(T(A), \mu_A)$ *of the category* $\mathbb{C}^{\mathbb{T}}$ *and that to each arrow* f *of the category* $\mathbb{C}$ *assigns the arrow* $T(f)$ *of the category* $\mathbb{C}^{\mathbb{T}}$.

Let $\mathbb{T}$ *be an object of the category* $\mathbf{WMon}(\mathbb{C})$.
For any $f \in \mathbb{C}^{\mathbb{T}}((T(A), \mu_A), (B, h))$*, we set*

$$\alpha^{\mathbb{T}}{}_{A,(B,h)}(f) = f \circ \eta_A \ .$$

For any $g \in \mathbb{C}(A, B)$ *and for any object* (B, h) *of* $\mathbb{C}^{\mathbb{T}}$*, we set*

$$\beta^{\mathbb{T}}{}_{A,(B,h)}(g) = h \circ T(g) \ .$$

Proposition 8 *The quadruple* $(F^{\mathbb{T}}, G^{\mathbb{T}}, \alpha^{\mathbb{T}}, \beta^{\mathbb{T}})$ *is a semi-adjunction* $\mathbb{C} \to \mathbb{C}^{\mathbb{T}}$.

Proof

Firstly, we note that for any $g \in \mathbb{C}(A,B)$ and for any object (B,h) of the category $\mathbb{C}^{\mathbb{T}}$, the diagram

$$\begin{tikzcd} T(A) \arrow[d, "h \circ T(g)"'] & T \circ T(A) \arrow[l, "\mu_A"'] \arrow[d, "T(h \circ T(g))"] \\ B & T(B) \arrow[l, "h"] \end{tikzcd}$$

commutes in the category $\mathbb{C}$. Hence we have ${\beta^{\mathbb{T}}}_{A,(B,h)}(g) \in \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h))$.

Let $j \in \mathbb{C}(A',A)$ andt $k \in \mathbb{C}^{\mathbb{T}}((B,h),(B',h'))$.

For any $f \in \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h))$, we have $k \circ f \circ T(j) \circ \eta_{A'} = k \circ f \circ \eta_A \circ j$, hence the diagram

$$\begin{tikzcd} \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h)) \arrow[r, "{\alpha^{\mathbb{T}}}_{A,(B,h)}"] \arrow[d, "k \circ - \circ F^{\mathbb{T}}(j)"'] & \mathbb{C}(A,B) \arrow[d, "G^{\mathbb{T}}(k) \circ - \circ j"] \\ \mathbb{C}^{\mathbb{T}}((T(A'),\mu_{A'}),(B',h')) \arrow[r, "{\alpha^{\mathbb{T}}}_{A',(B',h')}"'] & \mathbb{C}(A',B') \end{tikzcd} .$$

commutes.

For any $g \in \mathbb{C}(A,B)$, $k \circ g \circ j = k \circ h \circ T(g) \circ T(j) \circ \eta_{A'}$, hence the diagram

$$\begin{tikzcd} \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h)) \arrow[d, "k \circ - \circ F^{\mathbb{T}}(j)"'] & \mathbb{C}(A,B) \arrow[l, "{\beta^{\mathbb{T}}}_{A,(B,h)}"'] \arrow[d, "G^{\mathbb{T}}(k) \circ - \circ j"] \\ \mathbb{C}^{\mathbb{T}}((T(A'),\mu_{A'}),(B',h')) \arrow[r, "{\alpha^{\mathbb{T}}}_{A',(B',h')}"'] & \mathbb{C}(A',B') \end{tikzcd} .$$

commutes.

For any $f \in \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h))$, we have $h' \circ T(k \circ f \circ \eta_A \circ j) = k \circ f \circ T(j)$, hence the diagram

$$\begin{tikzcd} \mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h)) \arrow[r, "{\alpha^{\mathbb{T}}}_{A,(B,h)}"] \arrow[d, "k \circ - \circ F^{\mathbb{T}}(j)"'] & \mathbb{C}(A,B) \arrow[d, "G^{\mathbb{T}}(k) \circ - \circ j"] \\ \mathbb{C}^{\mathbb{T}}((T(A'),\mu_{A'}),(B',h')) & \mathbb{C}(A',B') \arrow[l, "{\beta^{\mathbb{T}}}_{A',(B',h')}"] \end{tikzcd} .$$

commutes.

For any $g \in \mathbb{C}(A,B)$, $k \circ h \circ T(g) \circ T(j) = h' \circ T(k \circ g \circ j)$, hence the diagram

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{T}}((T(A),\mu_A),(B,h)) \arrow[d, "k \circ - \circ F^{\mathbb{T}}(j)"'] & \mathbb{C}(A,B) \arrow[l, "{\beta^{\mathbb{T}}}_{A,(B,h)}"'] \arrow[d, "G^{\mathbb{T}}(k) \circ - \circ j"] \\
\mathbb{C}^{\mathbb{T}}((T(A'),\mu_{A'}),(B',h')) & \mathbb{C}(A',B') \arrow[l, "{\beta^{\mathbb{T}}}_{A',(B',h')}"]
\end{tikzcd} .$$

commutes.

We note that the induced weak monad by the semi-adjunction of the precedent proposition is $\mathbb{T}$. More precisely, the quadruple $(F^{\mathbb{T}}, G^{\mathbb{T}}, \alpha^{\mathbb{T}}, \beta^{\mathbb{T}})$ is a resolution of $\mathbb{T}$.

6 Some examples

We give some examples of weak comonads that provide (with some added structure) new denotational semantics of Linear Logic.

6.1 The relational weak comonad of finite sequences

This example is inspired by the tensor bialgebra.

For any set A, we set $T^1(A) = A^{<\omega}$, the set of finite sequences of elements of A. For any $f \subseteq A \times B$, $T^1(f)$ is the arrow of the category **Rel** *from $T^1(A)$ to $T^1(B)$ defined by setting*

$$T^1(f) \quad = \quad \left\{ \begin{array}{ll} (\langle \alpha_1, \ldots, \alpha_n \rangle, \langle \beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)} \rangle) \, ; & n \in \mathbb{N}, \\ & \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f) \\ & \text{and } \sigma \in \mathfrak{S}_n \end{array} \right\} .$$

For any set A, we define the arrow $\delta^1_A : T^1(A) \to (T^1 \circ T^1)(A)$ of the category **Rel** *by setting*

$$\delta^1_A = \{(a, \langle a_1, \ldots, a_k \rangle) \, ; \, k \in \mathbb{N}, \, a_1, \ldots, a_k \in A^{<\omega} \text{ and } a \in \mathfrak{S}(a_1, \ldots, a_k)\},$$

where $\mathfrak{S}(a_1, \ldots, a_k)$ is the set of shuffles of $a_1, \ldots, a_k$, i. e.

$$\begin{array}{c} \mathfrak{S}(a_1, \ldots, a_k) \\ = \left\{ \begin{array}{ll} \langle a_1, \ldots, a_n \rangle \, ; & \exists \sigma \in \mathfrak{S}_{p_1, \ldots, p_k} \\ & \forall i \in \mathbb{N} \, (1 \leq i \leq k \Rightarrow a_i = \langle \alpha_{\sigma(\sum_{j=1}^{i-1} p_j + 1)}, \ldots, \alpha_{\sigma(\sum_{j=1}^{i} p_j)} \rangle) \end{array} \right\} , \end{array}$$

with $\mathfrak{S}_{p_1, \ldots, p_k}$ the set of permutations $\sigma \in \mathfrak{S}_{\sum_{j=1}^{k} p_j}$ such that, for $1 \leq i \leq k$, σ is increasing on the interval $[(\sum_{j=1}^{i-1} p_j) + 1, \sum_{j=1}^{i} p_j]$.

For any set A, we define the arrow $d^1_A : T^1(A) \to A$ of the category **Rel** *by setting*

$$d^1_A = \{(\langle \alpha \rangle, \alpha) \, ; \, \alpha \in A\} \; .$$

We have: the triple (T^1, δ^1, d^1) is a weak comonad in the category **Rel**.

6.2 The relational weak comonad of labeled trees

This example is inspired by the Hopf algebra associated with the family of labeled trees ([7]). Let e be a set that is not a pair.

For any set A, for any integer n, we define by induction on n the set $\mathcal{CLT}_n(A)$ as follows:

- $\mathcal{CLT}_0(A) = \{(e, \alpha) \,;\, \alpha \in A\}$;
- $\mathcal{CLT}_{n+1}(A) = \mathcal{CLT}_n(A) \cup \{((e, \alpha), U) \,;\, \alpha \in A \text{ and } U \in \mathcal{M}_f(\mathcal{CLT}_n(A))\}$.

The set $\mathcal{CLT}(A)$ of completely labeled trees in A *is defined by setting*

$$\mathcal{CLT}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{CLT}_n(A) \ .$$

The set $\mathcal{LT}(A)$ of labelled trees in A *is defined as follows:*

$$\mathcal{LT}(A) = \{(e, U) \,;\, U \in \mathcal{M}_f(\mathcal{CLT}(A))\} \ .$$

For any set A, we define the function Node from $\mathcal{CLT}(A)$ to $\mathcal{M}_f(A)$ as follows:

- $\mathsf{Node}(e, \alpha) = [\alpha]$;
- $\mathsf{Node}((e, \alpha), [t_1, \ldots, t_n]) = [\alpha] + \sum_{i=1}^{n} \mathsf{Node}(t_i)$.

For any $a \in \mathcal{M}_f(A)$, we set

$$\mathcal{LT}(a) = \{(e, [t_1, \ldots, t_n]) \in \mathcal{LT}(A) \,;\, \sum_{i=1}^{n} \mathsf{Node}(t_i) = a\} \ .$$

For any set A, we set $T(A) = \mathcal{LT}(A)$. For any $f \subseteq A \times B$, $T^2(f)$ is the arrow $T^2(A) \to T^2(B)$ of the category **Rel** *by setting*

$$\begin{aligned} T^2(f) = \{ \quad & ((e, [(e, \alpha_1), \ldots, (e, \alpha_n)]), t) \,;\, \exists \beta_1, \ldots, \beta_n \\ & (\forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f) \text{ and } t \in \mathcal{LT}([\beta_1, \ldots, \beta_n]))\} \ . \end{aligned}$$

For any set A, we define the arrow $\delta^2_A : T^2(A) \to (T^2 \circ T^2)(A)$ of the category **Rel** *by setting*

$$\begin{aligned} \delta^2_A = \{ \quad & ((e, \textstyle\sum_{j=1}^{k} U_j), T) \,;\, k \in \mathbb{N}, U_1, \ldots, U_k \in \mathcal{M}_f(\mathcal{CLT}(A)) \\ & \text{and } T \in \mathcal{LT}([(e, U_1), \ldots, (e, U_k)])\} \ . \end{aligned}$$

For any set A, we define the arrow $d^2_A : T^2(A) \to A$ of the category **Rel** *by setting*

$$d^2_A = \{((e, [(e, \alpha)]), \alpha) \,;\, \alpha \in A\} \ .$$

We have: the triple (T^2, δ^2, d^2) is a weak comonad in the category **Rel**.

6.3

The relational comonad of finite multisets is well known. We denote by (T^0, δ^0, d^0) this comonad in the category **Rel**:

- *for any object A, $T^0(A) = \mathcal{M}_f(A)$ and, for any $f \in \mathbf{Rel}(A, B)$,*

$$T^0(f) = \{([\alpha_1, \ldots, \alpha_n], [\beta_1, \ldots, \beta_n]) \,;\, \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f)\} \ ;$$

- *for any object* A*,* $\delta_A \in \mathbf{Rel}(T(A), T \circ T(A))$ *defined by setting*

$$\delta^0{}_A = \{(\sum_{i=1}^{n} a_i, [a_1, \ldots, a_n]) \; ; \; a_1, \ldots, a_n \in T(A)\} \;\; ;$$

- *for any object* A*,* $d_A \in \mathbf{Rel}(T(A), A)$ *defined by setting*

$$d^0{}_A = \{([\alpha], \alpha) \; ; \; \alpha \in A\} \;\; .$$

Now, we can build new weak comonads using this comonad and the weak comonads of the precedent section. Let z *be a function that to each object of the category* $\mathbf{Rel}$ *assigns an element of* $\{0, 1, 2\}$*. For any object* A *of the category* $\mathbf{Rel}$*, we set* $T(A) = T^{z(A)}(A)$*,* $\delta_A = \delta^{z(A)}{}_A$ *and* $d_A = d^{z(A)}{}_A$*. For any arrow* $f : A \to B$*, we define* $T(f)$ *by setting:*

- *if* $z(A) = z(B)$*, then* $T(f) = T^{z(A)}(f)$*;*
- *if* $z(A) = 0$ *and* $z(B) = 1$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{([\alpha_1, \ldots, \alpha_n], \langle \beta_1, \ldots, \beta_n \rangle) \; ; \; \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f)\} \;\; ; \end{array}$$

- *if* $z(A) = 0$ *and* $z(B) = 2$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{([\alpha_1, \ldots, \alpha_n], t) \; ; \; \exists \beta_1, \ldots, \beta_n \\ & (\forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f) \text{ and } t \in \mathcal{LT}([\beta_1, \ldots, \beta_n]))\} \;\; ; \end{array}$$

- *if* $z(A) = 1$ *and* $z(B) = 0$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{(\langle \alpha_1, \ldots, \alpha_n \rangle, [\beta_1, \ldots, \beta_n]) \; ; \; \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f)\} \;\; ; \end{array}$$

- *if* $z(A) = 1$ *and* $z(B) = 2$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{(\langle \alpha_1, \ldots, \alpha_n \rangle, t) \; ; \; \exists \beta_1, \ldots, \beta_n \\ & (\forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f) \text{ and } t \in \mathcal{LT}([\beta_1, \ldots, \beta_n]))\} \;\; ; \end{array}$$

- *if* $z(A) = 2$ *and* $z(B) = 0$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{((e, [(e, \alpha_1), \ldots, (e, \alpha_n)]), [\beta_1, \ldots, \beta_n]) \; ; \; \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f)\} \;\; ; \end{array}$$

- *if* $z(A) = 2$ *and* $z(B) = 1$*, then*

$$\begin{array}{ll} & T(f) \\ = & \{ \; ((e, [(e, \alpha_1), \ldots, (e, \alpha_n)]), \langle \beta_1, \ldots, \beta_n \rangle) \; ; \quad \forall i \, (1 \leq i \leq n \Rightarrow (\alpha_i, \beta_i) \in f) \; \} \;\; . \end{array}$$

We have: the triple (T, δ, d) *is a weak comonad in the category* $\mathbf{Rel}$*.*

This fact follows from Theorem 1, that results from the study of denotational semantics of differential nets and of Linear Logic. Differential nets have been introduced in [5].

7 Taylor's Formula

We denote by $(\mathbf{Rel}, \otimes, I, \alpha, \lambda, \rho)$ *the following monoidal category. The functor* $\otimes : \mathbf{Rel} \times \mathbf{Rel} \to \mathbf{Rel}$ *is defined as follows: on objects, it is the cartesian product, on arrows, we have* $((\alpha, \beta), (\gamma, \delta)) \in f \otimes g$ *if, and only if,* $(\alpha, \gamma) \in f$ *and* $(\beta, \delta) \in g$*. The tensor unit* I *is the singleton.*

For any objects A *and* B *of the category* $\mathbf{Rel}$*,* 0_A^B *is the arrow* $A \to B$ *defined by setting* $0_A^B = \emptyset$*. If* $f, g \in \mathbf{Rel}(A, B)$*,* $f + g \in \mathbf{Rel}(A, B)$ *defined by* $f + g = f \cup g$*.*

Definition 28 *A relational model of differential nets is a family* $(T_A)_{A \in ob(\mathbf{Rel})}$ *with* $T_A = (T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A, cod_A, d_A)$*, where* $(T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A)$ *is a bialgebra in the category* $\mathbf{Rel}$*,* $cod_A \in \mathbf{Rel}(A, T(A))$*,* $d_A \in \mathbf{Rel}(T(A), A)$ *such that the diagrams*

$$\begin{tikzcd} A \arrow[r, "cod_A"] \arrow[dr, "0_A^I"'] & T(A) \arrow[d, "\epsilon_A"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} T(A) \arrow[r, "d_A"] & A \\ I \arrow[u, "\eta_A"] \arrow[ur, "0_I^A"'] & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "cod_A"] \arrow[dr, "((\eta_A \oplus cod_A) \circ \lambda_A{}^{-1}) \times ((cod_A \oplus \eta_A) \circ \rho_A{}^{-1})"'] & T(A) \arrow[d, "\Delta_A"] \\ & T(A) \otimes T(A) \end{tikzcd} \qquad \begin{tikzcd} T(A) \arrow[r, "d_A"] & A \\ T(A) \otimes T(A) \arrow[u, "\mu_A"] \arrow[ur, "(\lambda_A \circ (\epsilon_A \oplus d_A)) \times (\rho_A \circ (d_A \oplus \epsilon_A))"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "cod_A"] \arrow[dr, "id_A"'] & T(A) \arrow[d, "d_A"] \\ & A \end{tikzcd}$$

commute in the category $\mathbf{Rel}$.

Definition 29 *A relational model* $(T_A)_{A \in ob(\mathbf{Rel})}$ *of differential nets is said to be* commutative *(resp.* cocommutative*) if for any object* A *of the category* $\mathbf{Rel}$*, the bialgebra* $(T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A)$ *is commutative (resp. cocommutative).*

Example 1 *For any object* A *of the category* $\mathbf{Rel}$*, we set*

- $T^0(A) = \mathcal{M}_f(A)$*;*

- $\mu^0_A = \{((a,b), a+b);\ a, b \in \mathcal{M}_f(A)\}$;
- $\eta^0_A = \{(*, [])\}$;
- $\Delta^0_A = \{(a+b, (a,b));\ a, b \in \mathcal{M}_f(A)\}$;
- $\epsilon^0_A = \{([], *)\}$;
- $cod^0_A = \{(\alpha, [\alpha]);\ \alpha \in A\}$;
- $d^0_A = \{([\alpha], \alpha);\ \alpha \in A\}$.

The family $(T^0(A), \mu^0_A, \eta^0_A, \Delta^0_A, \epsilon^0_A, cod^0_A, d^0_A)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ is a relational model of differential nets, that is cocommutative and commutative.

Example 2 *For any object A of the category $\mathbf{Rel}$, we set*

- $T^1(A) = A^{<\omega}$;
- $\mu^1_A = \left\{ \begin{array}{ll} ((\langle \alpha_1, \ldots, \alpha_m \rangle, \langle \alpha_{m+1}, \ldots, \alpha_{m+n} \rangle), \langle \alpha_1, \ldots, \alpha_{m+n} \rangle); & m, n \in \mathbb{N} \textit{ and} \\ & \alpha_1, \ldots, \alpha_{m+n} \in A \end{array} \right\}$;
- $\eta^1_A = \{(*, \langle\rangle)\}$;
- $\Delta^1_A = \{(a, (a_1, a_2));\ a \in \mathfrak{S}(a_1, a_2)\}$;
- $\epsilon^1_A = \{(\langle\rangle, *)\}$;
- $cod^1_A = \{(\alpha, \langle \alpha \rangle);\ \alpha \in A\}$;
- $d^1_A = \{(\langle \alpha \rangle, \alpha);\ \alpha \in A\}$.

The family $(T^1(A), \mu^1_A, \eta^1_A, \Delta^1_A, \epsilon^1_A, cod^1_A, d^1_A)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ is a relational model of differential nets, that is cocommutative but non commutative.

Example 3 *For any $t \in \mathcal{CLT}(A)$, we set $depth(t) = Min\{n \in \mathbb{N};\ t \in \mathcal{CLT}_n(A)\}$.*

For any $t \in \mathcal{CLT}(A)$, for any $U_1, \ldots, U_n \in \mathcal{M}_f(\mathcal{CLT}(A))$, we define, by induction on $depth(t)$, $[U_1, \ldots, U_n] \cdot t \subseteq \mathcal{CLT}(A)$:

- $[U_1, \ldots, U_n] \cdot (e, \alpha) = \left\{ \begin{array}{ll} \emptyset & \textit{if } n \neq 1; \\ ((e, \alpha), U_1) & \textit{else;} \end{array} \right.$
- $$\begin{array}{l} [U_1, \ldots, U_n] \cdot ((e, \alpha), [t_1, \ldots, t_k]) \\ = \left\{ \begin{array}{ll} ((e, \alpha), U_{i_0} + [t'_1, \ldots, t'_k]); & i_0 \in \{1, \ldots, n\} \textit{ and} \\ & \exists \mathcal{U}_1, \ldots, \mathcal{U}_k \in \mathcal{M}_f(\mathcal{M}_f(\mathcal{CLT}(A))) \\ & ([U_{i_0}] + \sum_{j=1}^k \mathcal{U}_j = [U_1, \ldots, U_n] \textit{ and} \\ & \forall j \in \{1, \ldots, k\}\ t'_j \in \mathcal{U}_j \cdot t_j) \end{array} \right\}. \end{array}$$

Now, for any object A of the category $\mathbf{Rel}$, we set

- $T^2(A) = \mathcal{LT}(A)$;
- $\mu^2_A = \left\{ \begin{array}{ll} (((e, U_1 + U_2), & (e, [t_1, \ldots, t_k])), \quad (e, U_1 + [t'_1, \ldots, t'_k])); \\ & U_1, U_2 \in \mathcal{M}_f(\mathcal{CLT}(A)),\ t_1, \ldots, t_k \in \mathcal{CLT}(A) \textit{ and} \\ & \exists \mathcal{U}_1, \ldots, \mathcal{U}_k \in \mathcal{M}_f(\mathcal{M}_f(\mathcal{CLT}(A))) \\ & (\sum_{j=1}^k \mathcal{U}_j = U_2 \textit{ and } \forall j \in \{1, \ldots, k\}\ t'_j \in \mathcal{U}_j \cdot t_j) \end{array} \right\}$;

- $\eta_A^2 = \{(*, (e, []))\}$;
- $\Delta_A^2 = \{((e, U+V), ((e, U), (e, V)));\ U, V \in \mathcal{M}_f(\mathcal{CLT}(A))\}$;
- $\epsilon_A^2 = \{((e, []), *)\}$;
- $cod_A^2 = \{(\alpha, (e, [(e, \alpha)]));\ \alpha \in A\}$;
- $d_A^2 = \{((e, [(e, \alpha)]), \alpha);\ \alpha \in A\}$.

The family $(T^2(A), \mu_A^2, \eta_A^2, \Delta_A^2, \epsilon_A^2, cod_A^2, d_A^2)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ is a relational model of differential nets, that is cocommutative but non commutative.

Example 4 *For any object A of the category $\mathbf{Rel}$, we set*

- $T^3(A) = A^{<\omega}$;
- $\mu_A^3 = \{((a_1, a_2), a);\ a \in \mathfrak{S}(a_1, a_2)\}$;
- $\eta_A^3 = \{(*, \langle\rangle)\}$;
- $\Delta_A^3 = \left\{ \begin{array}{ll} (\langle\alpha_1, \ldots, \alpha_{m+n}\rangle, (\langle\alpha_1, \ldots, \alpha_m\rangle, \langle\alpha_{m+1}, \ldots, \alpha_{m+n}\rangle)); & m, n \in \mathbb{N} \text{ and} \\ & \alpha_1, \ldots, \alpha_{m+n} \in A \end{array} \right\}$;
- $\epsilon_A^3 = \{(\langle\rangle, *)\}$;
- $cod_A^3 = \{(\alpha, \langle\alpha\rangle);\ \alpha \in A\}$;
- $d_A^3 = \{(\langle\alpha\rangle, \alpha);\ \alpha \in A\}$.

The family $(T^3(A), \mu_A^3, \eta_A^3, \Delta_A^3, \epsilon_A^3, cod_A^3, d_A^3)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ is a relational model of differential nets, that is commutative but non cocommutative.

Example 5 *Let z be a function that to each object of the category $\mathbf{Rel}$ assigns an element of $\{0, 1, 2, 3\}$. For any object A of the category $\mathbf{Rel}$, we set $T(A) = T^{z(A)}(A)$, $\mu_A = \mu_A^{z(A)}$, $\eta_A = \eta_A^{z(A)}$, $\Delta_A = \Delta_A^{z(A)}$, $\epsilon_A = \epsilon_A^{z(A)}$, $cod_A = cod_A^{z(A)}$ and $d_A = d_A^{z(A)}$.*

We have: the family $(T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A, cod_A, d_A)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ is a relational model of differential nets. Moreover, if $Im(z) \subseteq \{0, 1, 2\}$, then the model is cocommutative.

Definition 30 *Given a relational model $(T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A, cod_A, d_A)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ of differential nets, for any $f \in \mathbf{Rel}(T(A), B)$, we define $f^\dagger \in \mathbf{Rel}(T(A), T(B))$ by setting*

$$f^\dagger = \bigcup_{n \in \mathbb{N}} (coc_B^n \circ \bigotimes_{i=1}^{n} (cod_B \circ f) \circ c_A^n) \ ,$$

where $coc_B^n \in \mathbf{Rel}(\bigotimes_{i=1}^n T(B), T(B))$ and $c_A^n \in \mathbf{Rel}(T(A), \bigotimes_{i=1}^n T(A))$ are defined by induction on n:

- *if $n = 0$, then $coc_B^n = \eta_B$ and $c_A^n = \epsilon_A$;*
- *if $n = 1$, then $coc_B^n = id_{T(B)}$ and $c_A^n = id_{T(A)}$;*
- *if $n \geq 1$, then $coc_B^{n+1} = \mu_B \circ (coc_B^n \otimes id_{T(B)})$ and $c_A^{n+1} = (c_A^n \otimes id_{T(A)}) \circ \Delta_A$.*

Theorem 1 *Let $\mathfrak{M} = (T(A), \mu_A, \eta_A, \Delta_A, \epsilon_A, cod_A, d_A)_{A \in \boldsymbol{ob}(\mathbf{Rel})}$ be a cocommutative relational model of differential nets.*

- *For any* $f \in \mathbf{Rel}(A,B)$*, we define* $T(f) \in \mathbf{Rel}(T(A),T(B))$ *by setting* $T(f) = (f \circ d_A)^\dagger$.
- *For any object* A *of the category* $\mathbf{Rel}$*, we define* $\delta_A \in \mathbf{Rel}(T(A), T \circ T(A))$ *by setting* $\delta_A = {id_{T(A)}}^\dagger$.

Then the triple $\mathbb{T} = (T, \delta, d)$ *is a weak comonad in the category* $\mathbf{Rel}$*. Moreover, if the model* $\mathfrak{M}$ *is not commutative,* $\mathbb{T}$ *is not a comonad.*

The proof of this theorem is given in [4]. In fact, there we proved that this construction provides more than a weak comonad: it provides a model of Linear Logic, i.e. it provides a categorical structure that satisfies our new axiomatics.

If we apply the theorem to Example 1 (resp. Example 2, resp. Example 3), we obtain the relational monad of finite multisets (resp. the weak monad of 6.1, resp. the weak monad of 6.2). If we apply the theorem to Example 5 with the restriction that we have $Im(z) \subseteq \{0,1,2\}$*, we obtain the weak monad of 6.3 with the the same function* z.

Note that the assumption of the cocommutativity in the theorem is necessary, even if we would assume that the relational model of differential nets is commutative; a counter-example is given by the family $(T^3(A), \mu^3_A, \eta^3_A, \Delta^3_A, \epsilon^3_A, cod^3_A, d^3_A)_{A \in \mathbf{ob}(\mathbf{Rel})}$*. Indeed, in this case, if* A *is any set that contains at least three elements* α_1*,* α_2 *and* α_3*, we have*

$$(\langle \alpha_1, \alpha_2, \alpha_3 \rangle, \langle \langle \alpha_1, \alpha_3 \rangle, \langle \alpha_2 \rangle \rangle) \in \delta_A \circ T(id_A)$$

and we have not

$$(\langle \alpha_1, \alpha_2, \alpha_3 \rangle, \langle \langle \alpha_1, \alpha_3 \rangle, \langle \alpha_2 \rangle \rangle) \in (T \circ T(id_A)) \circ \delta_A \quad .$$

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